Loop Objects in Pointed Derivators

Aras Ergus

Geboren am 19. Mai 1993 in Osmangazi

29. Juni 2015

Bachelorarbeit Mathematik

Betreuer: Dr. Moritz Groth

Zweitgutachter: Prof. Dr. Stefan Schwede

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Zusammenfassung

Ein wichtiger Ansatz in der Homotopietheorie (von topologischen Räumen, Kettenkomplexen usw.) ist das Studium von *Homotopie(ko)limetes* bzw. allgemeiner von *Homotopiekanerweiterungen*. Diese verallgemeinern viele homotopietheoretische Konstruktionen und haben nützliche Eigenschaften, die ähnlich zu denen von klassischen Kanerweiterungen sind. *Derivatore* axiomatisieren diesen Kalkül von (Homotopie-)Kanerweiterungen und erlauben uns somit, mit abstrakten Mitteln Homotopietheorie zu betreiben.

Wir betrachten zum Beispiel den Schleifenraum $\Omega(X, x)$ eines punktierten topologischen Raumes (X, x). Dieser kann als das Homotopiepullback des Diagramms

$$\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & (X, x) \end{array}$$

aufgefasst werden, wobei $*$ ein einpunktiger Raum ist. Außerdem besitzt $\Omega(X, x)$ in der Homotopiekategorie von punktierten Räumen die Struktur eines Gruppenobjekts, was durch Verkettung und Umkehrung von Schleifen gegeben ist.

Es gibt in der Tat analoge Konstruktionen für „punktierte Derivatore“. In dieser Arbeit geht es um diese „Schleifenobjekte“ in Werten von punktierten Derivatoren. Sie beginnt mit einer Zusammenfassung von wichtigen Definitionen und Resultaten aus der Theorie von Derivatoren. In dem ersten Abschnitt des Hauptteils wird dann eine Gruppenobjektstruktur für Schleifenobjekte konstruiert. Daraufhin werden im zweiten Abschnitt zweifache Schleifenobjekte untersucht, die sogar eine abelsche Gruppenobjektstruktur besitzen. Der Hauptteil endet mit einem kurzen Abschnitt über Anwendungen, in dem Resultate über stabile bzw. verschobene Derivatore bewiesen werden.

Die Arbeit enthält außerdem zwei Anhänge. Der erste Anhang behandelt eine alternative Darstellung von Monoidobjekten mit Hilfe von der Segal-Bedingung, die eine wesentliche Rolle bei der Konstruktion der Gruppenobjektstruktur von Schleifenobjekten spielt. In dem zweiten Anhang geht es um alternative Charakterisierungen von (prä)additiven Kategorien, die in der Untersuchung von stabilen Derivatoren verwendet werden.

Contents

Introduction **7**

0 Preliminaries **9**
Conventions and Notations 9
A Review of Derivators 10

1 Loop Objects **17**
Simplicial Objects which Induce Loop Objects 17
Loop Objects as Monoid Objects 19
Loop Objects as Group Objects 24

2 Double Loop Objects **26**
Loop Functor as a Functor to Group Objects 26
Products under the Loop Functor 26
The Eckmann–Hilton Argument 28

3 Applications **31**

Appendix A The Segal Condition **33**

Appendix B Additive Categories **40**

References **47**

Introduction

Motivation

One way of studying homotopy theory (of topological spaces, chain complexes etc.) is considering so-called *homotopy (co)limits* or more generally *homotopy Kan extensions*. Besides encompassing many homotopy theoretical constructions, these have very useful formal properties similar to the properties of classical Kan extensions. The concept of a *derivator* provides an abstract framework for homotopy theory by axiomatizing this calculus of (homotopy) Kan extensions.

For example, consider the loop space $\Omega(X, x)$ of a given pointed topological space (X, x). This can be thought of as the homotopy pullback of the diagram

$$\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & (X, x) \end{array} ,$$

where $*$ is a space with only one point. Moreover, note that in the homotopy category of pointed spaces, $\Omega(X, x)$ has a group object structure given by concatenation and inversion of loops.

There are indeed analogous constructions in the setting of so-called "pointed derivators". Furthermore, one can show that these "loop objects" have a canonical group object structure, which is the main topic of this thesis.

About This Thesis

The thesis starts with a review of derivators. It is by far not a detailed introduction to the theory of derivators, but merely a summary of some results which are needed later.

Having all required concepts at hand, the first section of the main part deals with a construction which yields a group object structure on loop objects. Here some generalities on simplicial objects are needed, which are covered in the first appendix.

From this point on, the technical details about derivators are not very important. One can, for instance, show that twofold loop objects have an abelian group object structure by using a formal Eckmann–Hilton argument. This is done in the second section of the main part.

The main part ends with a short section on applications. The main application is the additivity of so-called "stable derivators". For this some general statements about additive categories are needed, which are dealt with in the second appendix. Furthermore, it is also mentioned that these results can be applied to not only the underlying category of a derivator, but to all of its values.

Even though the introduction is written from a first person perspective, the "mathematical we" will accompany the reader in the main part of the thesis.

Acknowledgments and Thanks

My knowledge of derivators is mostly based on an introductory course at the University of Bonn which eventually led to this thesis. I would like to thank my advisor Dr. Moritz Groth for giving this course and supervising my thesis.

A significant part of this thesis was written in the recreation room at the Mathematical Institute. I would like to thank my fellow students for creating such a productive environment in this room.

Last but not least, I would like to thank my parents and my brother for supporting me throughout my studies.

0 Preliminaries

Conventions and Notations

When we consider a *category*, it is mostly implicit that the category in consideration is *locally small*, i. e. the homomorphism classes are sets. In a few situations we will make this explicit in order to emphasize that the homomorphism classes are indeed sets.

There are two cases, namely Definition 0.6 and Lemma 1.3, where we talk about "functors" into "categories" which are not necessarily locally small. This is also emphasized by the usage of capital letters in notations like **CAT** and END. In these cases, one must indeed confront some "size issues" which we do not want to deal with in detail. Our approach is considering these statements as "metatheoretical" ones and explicitly writing down what they mean.

Moreover, the diagrams we consider do not necessarily commute unless explicitly stated.

Now we introduce some concepts and notations which will be used throughout the thesis.

Notation 0.1. Let $\circledast$ denote the category which has a unique object $*$ with $\mathrm{End}_{\circledast}(*) = \mathrm{id}_*$, which is a terminal object in the category **Cat** of small categories.

Notation 0.2. Let $\mathscr{C}$ be a category and A a small category. Then $\mathscr{C}^A$ will denote the category of functors $A \to \mathscr{C}$ with natural transformations between such functors as morphisms.

Note that in this case $\mathscr{C}^A$ is indeed a locally small category since the class of natural transformations between two functors $F, G\colon A \to \mathscr{C}$ can essentially be seen as a subclass of $\prod_{a\in \mathrm{ob}\, A} \mathrm{Hom}_{\mathscr{C}}(F(a), G(a))$ which is a set.

Notation 0.3. Let $\mathscr{C}$ be a category which has products and coproducts.

For a family $(X_i)_{i\in I}$ of objects in $\mathscr{C}$ we will denote the structure morphisms of the product by $\mathrm{pr}_i\colon \prod_{i\in I} X_i \to X_i$. Given a family of morphisms $(f_i\colon Y \to X_i)_{i\in I}$ in $\mathscr{C}$, we will denote the induced morphism to the product by $\prod_{i\in I} f_i\colon Y \to \prod_{i\in I} X_i$. If $I = \{1, \ldots, n\}$ is finite, the product will alternatively be denoted by $X_1 \times \cdots \times X_n$ and the morphism into the product which is induced by the family $(f_1, \ldots, f_n)$ by $f_1 \times \cdots \times f_n$.

Similarly, we will write $\mathrm{in}_i\colon X_i \to \coprod_{i\in I} X_i$ for the structure morphisms of a coproduct and $\coprod_{i\in I} g_i\colon \coprod_{i\in I} X_i \to Y$ for the morphism from the coproduct which is induced by the family $(g_i\colon X_i \to Y)_{i\in I}$ of morphisms. In the finite case $X_1 \amalg \cdots \amalg X_n$ resp. $g_1 \amalg \cdots \amalg g_n$ will be the alternative notation.

Notation 0.4. If a category $\mathscr{C}$ has a terminal object, we will denote it usually by $\circledast$. Note that this is compatible with Notation 0.1 for **Cat**.

Furthermore, this notation will not cause any ambiguities since it will be clear from context which category $*$ belongs to.

Given an object $X \in \operatorname{ob} \mathscr{C}$, π_X will denote the unique morphism $X \to *$.

Definition 0.5. If a category $\mathscr{C}$ has an object which initial and terminal, we call $\mathscr{C}$ *pointed.* Such an object is also called a *zero object* and usually denoted by 0.

A Review of Derivators

Derivators are a way of axiomatizing (homotopy) Kan extensions and hence allow us to describe homotopy theories in purely (2-)categorical terms. Their theory was first developed by Heller (cf. [7]) and Grothendieck (cf. [6]). Later on, the stable case was studied independently by Franke (cf. [3]).

In this subsection, we will briefly review the parts of this theory which we will need in the following sections and try to make some of our slogans into more precise statements. While doing this, we will follow the approach in [4] and [5], where most of the omitted proofs can be found.

We start with the definition of the underlying data of a derivator.

Definition 0.6. A *prederivator* $\mathscr{D}$ is a strict 2-functor from the category **Cat** of small categories to the category **CAT** of all categories which is contravariant on functors (and covariant on natural transformations).

This means that $\mathscr{D}$ assigns to each small category A a category $\mathscr{D}(A)$, to each functor $u\colon A \to B$ between small categories a functor $\mathscr{D}(u)\colon \mathscr{D}(B) \to \mathscr{D}(A)$ and to each natural transformation $\rho\colon u \Rightarrow v$ between such functors a natural transformation $\mathscr{D}(\rho)\colon \mathscr{D}(u) \Rightarrow \mathscr{D}(v)$ such that everything is compatible with compositions and identities.

For the rest of this section $\mathscr{D}$ will be a prederivator. Furthermore, throughout the thesis we will use a shorter notation for the value of a functor under a prederivator.

Notation 0.7. Given a functor $u\colon A \to B$ between small categories, we will write u^* for $\mathscr{D}(u)\colon \mathscr{D}(B) \to \mathscr{D}(A)$ as long as the prederivator in consideration is clear from context.

Here the value $\mathscr{D}(A)$ (resp. $\mathscr{D}(B)$) can be thought of as the category of “coherent diagrams of shape A (resp. B)” and the induced functor $u^*\colon \mathscr{D}(B) \to \mathscr{D}(A)$ as a “restriction functor” (cf. Example 0.14 and Example 0.15). Indeed, if one considers $\mathscr{D}(*)$ as “the underlying category of $\mathscr{D}$”, one can construct actual diagrams from coherent diagrams in the following sense.

Remark 0.8. Given a category $\mathscr{C}$, an object $C \in \operatorname{ob} \mathscr{C}$ corresponds to the functor $* \to \mathscr{C}$ which sends $*$ to C and id_* to id_C. Under this identification a natural transformation between two functors $C, D\colon \ast \to \mathscr{C}$ corresponds to a morphism $C(\ast) \to D(\ast)$. Furthermore, this assignment induces an isomorphism of categories between $\mathscr{C}$ and $\mathscr{C}^{\ast}$.

Hence, given a small category A and an object a of A, one can consider a as a functor $a\colon \ast \to A$. This yields a functor $a^*\colon \mathscr{D}(A) \to \mathscr{D}(\ast)$. We will denote it also by $(\_)_a$ and hence write $f_a\colon X_a \to Y_a$ for the image of a morphism $f\colon X \to Y$ in $\mathscr{D}(A)$ under a^*.

Moreover, morphisms $\varphi\colon a \to a'$ in A correspond to natural transformations $\varphi\colon a \Rightarrow a'$. Hence a morphism in A yields a natural transformation $a^* \Rightarrow (a')^*$ between Functors $a^*, (a')^*\colon \mathscr{D}(A) \to \mathscr{D}(\ast)$. In fact, to each $X \in \operatorname{ob}\mathscr{D}(A)$ we can (functorially) assign an object $\mathrm{dia}_A(X)$ of $\mathscr{D}(\ast)^A$ by setting $\mathrm{dia}_A(X)(a) := X_a$ for objects and $\mathrm{dia}_A(X)(\varphi) := \mathscr{D}(\varphi)_X\colon X_a \to X_{a'}$ for morphisms. We call $\mathrm{dia}_A(X)$ the *underlying diagram of* X.

The next step is imitating the description of (homotopy) Kan extensions as adjoints to restriction functors.

Definition 0.9. Let $u\colon A \to B$ be a functor between small categories.

- We say that $\mathscr{D}$ *admits left Kan extensions along* u if u^* has a left adjoint $u_!$. In this case we denote the unit of this adjunction by $\eta_{u_!,u^*}$ and the counit by $\epsilon_{u_!,u^*}$.
- We say that $\mathscr{D}$ *admits right Kan extensions along* u if u^* has a right adjoint u_*. In this case we denote the unit of this adjunction by η_{u^*,u_*} and the counit by ϵ_{u^*,u_*}.

In order to make the definition of derivators more understandable, we will first review the calculus of "mates" which plays a central role in the theory derivators.

Definition 0.10. Given functors $A \xrightarrow{p} B \xrightarrow{u} D$ and $A \xrightarrow{v} C \xrightarrow{q} D$ between small categories and a natural transformation $u \circ p \Rightarrow q \circ v$, applying $\mathscr{D}$ yields a diagram

$$\begin{array}{ccc} \mathscr{D}(A) & \xleftarrow{p^*} & \mathscr{D}(B) \\ {\scriptstyle v^*}\uparrow & \swarrow & \uparrow{\scriptstyle u^*} \\ \mathscr{D}(C) & \xleftarrow[q^*]{} & \mathscr{D}(D) \end{array}$$

with a natural transformation $\alpha\colon p^* \circ u^* \Rightarrow v^* \circ q^*$.

Assuming that $\mathscr{D}$ admits both left and right Kan extensions along all functors involved, the corresponding adjunction units and counits yield two diagrams

$$\begin{array}{ccccccc} \mathscr{D}(C) & \xleftarrow{v_!} & \mathscr{D}(A) & \xleftarrow{p^*} & \mathscr{D}(B) & \xleftarrow{\mathrm{id}_{\mathscr{D}(B)}} & \\ & \swarrow & {\scriptstyle v^*}\uparrow & \swarrow & \uparrow{\scriptstyle u^*} & \swarrow & \\ \mathrm{id}_{\mathscr{D}(C)} & & \mathscr{D}(C) & \xleftarrow[q^*]{} & \mathscr{D}(D) & \xleftarrow[u_!]{} & \mathscr{D}(B) \end{array}$$

and

$$\begin{array}{ccc}
\mathscr{D}(B) & \xleftarrow{\quad\mathrm{id}_{\mathscr{D}(B)}\quad} & \\
{\scriptstyle p_*}\uparrow & \Swarrow & \\
\mathscr{D}(A) & \xleftarrow{\;p^*\;} & \mathscr{D}(B) \\
{\scriptstyle v^*}\uparrow & \Swarrow & \uparrow{\scriptstyle u^*} \\
\mathscr{D}(C) & \xleftarrow[\;q^*\;]{} & \mathscr{D}(D) \\
 & \Swarrow & \uparrow{\scriptstyle q_*} \\
 & \xleftarrow[\quad\mathrm{id}_{\mathscr{D}(C)}\quad]{} & \mathscr{D}(C)
\end{array} \; .$$

Hence we obtain natural transformations

$$\alpha_! \colon v_! \circ p^* \xRightarrow{(v_! \circ p^*)\,\eta_{u_!,u^*}} v_! \circ p^* \circ u^* \circ u_! \xRightarrow{v_!\,\alpha\,u_!} v_! \circ v^* \circ q^* \circ u_! \xRightarrow{\epsilon_{v_!,v_*}\,(q^* \circ u_!)} q^* \circ u_!$$

and

$$\alpha_* \colon u^* \circ q_* \xRightarrow{\eta_{p^*,p_*}\,(u^* \circ q_*)} p_* \circ p^* \circ u^* \circ q_* \xRightarrow{p_*\,\alpha\,q_*} p_* \circ v^* \circ q^* \circ q_* \xRightarrow{(p_* \circ v^*)\,\epsilon_{q^*,q_*}} p_* \circ v^*$$

which we call the *mate transformations associated to* α.

We now summarize some important results about mates.

Remark 0.11. In the situation of the previous definition the following statements hold.

- Consider the "degenerate" case where α is induced by a commutative square

  $$\begin{array}{ccc}
  A & \xrightarrow{\mathrm{id}_A} & A \\
  {\scriptstyle v}\downarrow & /\!\!/ & \downarrow{\scriptstyle v} \\
  C & \xrightarrow[\mathrm{id}_C]{} & C,
  \end{array}$$

  i.e. the natural transformation $\mathrm{id}_v \colon v \Rightarrow v$. Then $\alpha = \mathrm{id}_{v_*}$ by the 2-functoriality of $\mathscr{D}$ and $\alpha_! = \mathrm{id}_{v_!} \colon v_! \Rightarrow v_!$ by the triangular identity for the left adjoint functor $v_!$.

- Similarly, if $A = C$, $B = D$, $v = \mathrm{id}_A$, $u = \mathrm{id}_B$, $p = q$ and the given natural transformation $p \Rightarrow p$ is id_p, then $\alpha_* = \mathrm{id}_{p_*}$.

- Consider a "horizontally adjacent square" as in

  $$\begin{array}{ccccc}
  A & \xrightarrow{p} & B & \xrightarrow{r} & E \\
  {\scriptstyle v}\downarrow & \Swarrow & \downarrow{\scriptstyle u} & \Swarrow & \downarrow{\scriptstyle t} \\
  C & \xrightarrow[q]{} & D & \xrightarrow[s]{} & F
  \end{array} ,$$

where we call the natural transformation induced by the new square $\beta\colon r^* \circ t^* \Rightarrow u^* \circ s^*$. Then we obtain a *horizontal pasting*

$$\gamma\colon p^* \circ r^* \circ t^* \xRightarrow{p^* \beta} p^* \circ u^* \circ s^* \xRightarrow{\alpha\, s^*} v^* \circ q^* \circ s^*.$$

Furthermore, $\gamma_!$ coincides with the horizontal pasting of $\alpha_!$ and $\beta_!$.

- Similarly, given a "vertically adjacent square" as in

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle v}\downarrow & \swArrow & \downarrow{\scriptstyle u} \\ C & \xrightarrow[q]{} & D \\ {\scriptstyle w}\downarrow & \swArrow & \downarrow{\scriptstyle t} \\ E & \xrightarrow{r} & F \end{array},$$

we obtain a *vertical pasting* $p^* \circ u^* \circ t^* \Rightarrow v^* \circ w^* \circ r^*$ which is compatible with $(\_)_*$.

- $\alpha_!$ is an isomorphism if and only if α_* is an isomorphism.

We need one more concept before we can give a definition of derivators.

Definition 0.12. Let $u\colon A \to B$ be a functor between small categories and $b \in \operatorname{ob} B$.

Then we obtain two *slice categories* (u/b) and (b/u) given by

$$\operatorname{ob}((u/b)) = \{(a, f) \mid a \in \operatorname{ob} A, f\colon u(a) \to b\}$$

and

$$\operatorname{Hom}_{(u/b)}\big((a, f), (a', f')\big) = \big\{g \in \operatorname{Hom}_A\big(a, a'\big) \mid f = f' \circ u(g)\big\}$$

respectively

$$\operatorname{ob}((b/u)) = \{(a, f) \mid a \in \operatorname{ob} A, f\colon b \to u(a)\}$$

and

$$\operatorname{Hom}_{(b/u)}\big((a, f), (a', f')\big) = \big\{g \in \operatorname{Hom}_A\big(a, a'\big) \mid f' = u(g) \circ f\big\}.$$

There is a functor $p\colon (u/b) \to A$ which is given by $p((a, f)) = a$ on objects and $p(g) = g$ on morphisms. Similarly, there is a functor $q\colon (b/u) \to A$ given by $q((a, f)) = a$ on objects and $q(g) = g$ on morphisms.

Furthermore, there is a natural transformation $\phi_{u,b}$ from $u \circ p$ to the constant functor $b \circ \pi_{(u/b)}$ that is given by $f\colon u(p((a, f))) = u(a) \to b$ for each $(a, f) \in \operatorname{ob}((u/b))$. Similarly, there is a natural transformation $\psi_{u,b}\colon b \circ \pi_{(b/u)} \Rightarrow u \circ q$ which is given by $f\colon b \to u(a) = u(q((a, f)))$ for $(a, f) \in \operatorname{ob}((u/b))$.

Now we have everything at hand to define derivators.

Definition 0.13. A prederivator $\mathscr{D}$ is called a *derivator* if the following conditions are satisfied:

(Der1) $\mathscr{D}$ takes coproducts to products, i.e. for a family $(A_i)_{i\in I}$ of small categories the functor

$$\mathscr{D}\left(\coprod_{i\in I} A_i\right) \xrightarrow{\prod_{i\in I} \mathrm{in}_i^*} \prod_{i\in I} \mathscr{D}(A_i)$$

is an equivalence of categories.

(Der2) For $A \in \mathrm{ob}\,\mathbf{Cat}$, a morphism $f\colon X \to Y$ in $\mathscr{D}(A)$ is an isomorphism iff for each $a \in \mathrm{ob}\,A$, $f_a\colon X_a \to Y_a$ is an isomorphism in $\mathscr{D}(\ast)$.

(Der3) For a functor $u\colon A \to B$ between small categories, $\mathscr{D}$ has both left and right Kan extensions along u.

(Der4) For a functor $u\colon A \to B$ between small categories and an object $b \in B$, the mate transformation

$$(\Phi_{u,b})_!\colon (\pi_{u/b})_! \circ p^* \Rightarrow b^* \circ u_!$$

induced by $\phi_{u,b}$ and the mate transformation

$$(\Psi_{u,b})_*\colon b^* \circ u_* \Rightarrow (\pi_{(b/u)})_* \circ q^*$$

induced by $\psi_{u,b}$ are isomorphisms.

Note that the axioms (Der1)-(Der4) do not add any new data to a prederivator $\mathscr{D}$, but they merely require $\mathscr{D}$ to have certain properties. Even (Der3) essentially does not require any choices to be made since adjoint functors are unique up to isomorphism if they exist.

Before we proceed with some useful properties of derivators, we want to consider some examples which are the main motivation for studying derivators. We start with a classical one which is essentially discussed in [9, Chapter X] without the terminology of derivators.

Example 0.14. To each category $\mathscr{C}$ we can assign its *represented prederivator* $y_{\mathscr{C}}$ which is given by

- $y_{\mathscr{C}}(A) = \mathscr{C}^A$ on small categories $A \in \mathrm{ob}\,\mathbf{Cat}$,
- $u^* = \_ \circ u\colon \mathscr{C}^B \to \mathscr{C}^A$ on functors $u\colon A \to B$ between small categories
- $(y_{\mathscr{C}}(\alpha))_F = F\alpha\colon F \circ u \Rightarrow F \circ v$ for $F \in \mathscr{C}^B$ on natural transformations $\alpha\colon u \Rightarrow v$ between such functors.

Note that represented prederivators always satisfy (Der1) and (Der2).

Now assume that $\mathscr{C}$ is complete and cocomplete. Then, for a functor $u\colon A \to B$, $u^*\colon \mathscr{C}^B \to \mathscr{C}^A$ has indeed both a left adjoint and a right adjoint, which are given by classical Kan extensions along u. This means that (Der3) holds. Furthermore, there are "formulas" for computing these classical Kan extensions pointwise in terms of colimits and limits which correspond to the axiom (Der4).

Another example which is more general and more closely related to homotopy theory is given by model categories. The statement in its full generality is due to Cisinski (cf. [1]).

Example 0.15. Let $\mathscr{M}$ be a model category. Given a small category A, we denote by W_A the class of morphisms in $\mathscr{M}^A$ which are pointwise weak equivalences.

Then one can construct a derivator $\mathscr{H}o_{\mathscr{M}}$, called the *homotopy derivator of* $\mathscr{M}$, which is given by localizations $\mathscr{M}^A[W_A^{-1}]$ (which can be realized as locally small categories) on objects $A \in \operatorname{ob}\mathbf{Cat}$ and restriction functors induced by $\_ \circ u\colon \mathscr{M}^B \to \mathscr{M}^A$ on functors $u\colon A \to B$.

In particular, one has homotopy derivators associated with chain complexes of modules over a ring, (pointed) topological spaces, (pointed) simplicial sets and spectra.

Let $\mathscr{D}$ be a derivator for the rest of this section. Then (Der1) and (Der3) yield the following statement.

Remark 0.16. As the diagonal functor $\Delta\colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)^I \simeq \mathscr{D}(I)$ has a left adjoint and a right adjoint for any index set I, coproducts and products exist in $\mathscr{D}(\ast)$.

In particular, $\mathscr{D}(\ast)$ has an initial object $\varnothing$ and a terminal object $\ast$ (which is not to be confused with the terminal category $\ast$).

There are certain types of functors $u\colon A \to B$ for which Kan extensions along u are particularly easy to understand.

Definition 0.17. Let $u\colon A \to B$ be a fully faithful functor between small categories.

- u is called a *cosieve* if for every morphism $u(a) \to b$ in B it follows that $b \in \operatorname{essim} u$.
- u is called a *sieve* if for every morphism $b \to u(a)$ in B it follows that $b \in \operatorname{essim} u$.

Indeed, some calculations yield the following very useful statement.

Remark 0.18. If $u\colon A \to B$ is a cosieve, then $X \in \mathrm{ob}(\mathscr{D}(B))$ is in the essential image of $u_!$ if and only if $X_b \cong \varnothing$ for all $b \in (\mathrm{ob}\, B)\backslash u(\mathrm{ob}\, A)$.

Similarly, if $u\colon A \to B$ is a sieve, then $X \in \mathrm{ob}(\mathscr{D}(B))$ is in the essential image of u_* if and only if $X_b \cong \ast$ for all $b \in (\mathrm{ob}\, B)\backslash u(\mathrm{ob}\, A)$.

There is one more relevant case in which Kan extensions can be computed easily.

Remark 0.19. Let A be a small category.

- If A has a terminal object ∞, then we have $(\pi_A)_!(X) \cong X_\infty$ for all $X \in \mathrm{ob}(\mathscr{D}(A))$.
- If A has an initial object $\varnothing$, then we have $(\pi_A)_*(X) \cong X_\varnothing$ for all $X \in \mathrm{ob}(\mathscr{D}(A))$.

We end this section with definitions of a few concepts which play an important role in the theory of derivators.

Definition 0.20. A square

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle v}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\ C & \xrightarrow[q]{} & D \end{array}$$

in **Cat** is called *homotopy exact*, if for any derivator $\mathscr{E}$, the mate transformations associated with

$$\begin{array}{ccc} \mathscr{E}(A) & \xleftarrow{p^*} & \mathscr{E}(B) \\ {\scriptstyle v^*}\uparrow & \overset{\alpha}{\swarrow} & \uparrow{\scriptstyle u^*} \\ \mathscr{E}(C) & \xleftarrow[q^*]{} & \mathscr{E}(D) \end{array}$$

are isomorphisms.

Note that in the situation of the previous definition, $\alpha_!$ is an isomorphism if and only if α_* is an isomorphism. In particular, (Der4) means that all "slice squares" of the form

$$\begin{array}{ccc} (u/b) & \xrightarrow{p} & A \\ {\scriptstyle \pi_{(u/b)}}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\ \ast & \xrightarrow[b]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} (b/u) & \xrightarrow{q} & A \\ {\scriptstyle \pi_{(b/u)}}\downarrow & \nearrow & \downarrow{\scriptstyle u} \\ \ast & \xrightarrow[b]{} & B \end{array}$$

are homotopy exact.

The last definition of this section brings us closer to the title of this thesis.

Definition 0.21. $\mathscr{D}$ is called *pointed* if $\mathscr{D}(\ast)$ has a zero object 0.

An important class of examples for pointed derivators is given by homotopy derivators of pointed model categories.

1 Loop Objects

The aim of this section is showing that loop objects in the underlying category of a pointed derivator have a group object structure. This is done by constructing a special simplicial object whose first level coincides with the loop object and then showing that there is an inversion morphism for the induced multiplication (cf. Definition A.5 and Proposition A.6).

Let $\mathscr{D}$ be a pointed derivator throughout this section.

Simplicial Objects which Induce Loop Objects

The crucial point of our discussion of the loop objects is the fact that they are induced by certain families of objects which have a richer structure than that of a simplicial object.

In order to define these objects, we need a few new notations.

Notation 1.1.

- Let **Fin** denote the category of finite sets (or equivalently the category of finite discrete categories). Let $\langle n \rangle := \{0, \ldots, n\} \in \operatorname{ob} \mathbf{Fin}$ for $n \in \mathbb{N}$.
- Let $(\_)^{\triangleright} \colon \mathbf{Fin} \to \mathbf{Cat}$ be the cocone functor, i. e. the functor which adds a terminal object ∞ to a given category. Let $\lrcorner_n := \langle n \rangle^{\triangleright}$.

Now we can define the structures which induce the loop objects.

Definition 1.2. For $a \in \operatorname{ob} \mathbf{Fin}$ let ω_a be the composition

$$\omega_a \colon \mathscr{D}(\ast) \xrightarrow{\infty_!} \mathscr{D}(a^{\triangleright}) \xrightarrow{(\pi_{a^{\triangleright}})_*} \mathscr{D}(\ast).$$

For $n \in \mathbb{N}$ we will abuse notation and write ω_n for $\omega_{\langle n \rangle}$. In particular, for the loop functor defined in [5, Definition 8.17] we have

$$\Omega \cong \omega_1 \colon \mathscr{D}(\ast) \xrightarrow{\infty_!} \mathscr{D}(\lrcorner_1) \xrightarrow{(\pi_{\lrcorner_1})_*} \mathscr{D}(\ast).$$

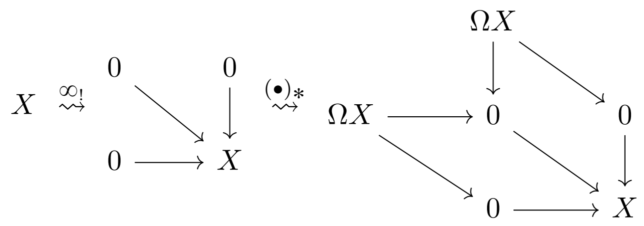
Before we investigate these objects further, let us have a look at how ω_n for small n behaves on the level of underlying diagrams. First we note that $\infty_! \colon \mathscr{D}(\ast) \to \mathscr{D}(\langle n \rangle^{\triangleright})$ simply "adds zeros" since $\infty \colon \ast \to \lrcorner_n$ is a cosieve (cf. Remark 0.18). In particular, for ω_0 we have

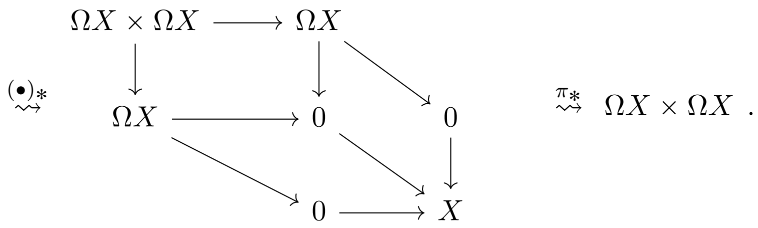
$$X \overset{\infty_!}{\rightsquigarrow} 0 \to X \overset{\pi_*}{\rightsquigarrow} 0 \ .$$

For higher degrees we can add some intermediate steps to have a better understanding and thus obtain

$$X \overset{\infty_!}{\rightsquigarrow} \begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0 & \longrightarrow & X \end{array} \overset{(\bullet)_*}{\rightsquigarrow} \begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array} \overset{\pi_*}{\rightsquigarrow} \Omega X$$

respectively





We will make these pictures into precise statements in the following pages. In particular, we will generalize the fact that $\omega_2 X$ is isomorphic to $\omega_1 X \times \omega_1 X$ and show that the Segal condition holds for $(\omega_n X)_{n \in \mathbb{N}}$. However, before we can talk about the Segal condition, we have to first show that our construction is functorial in n.

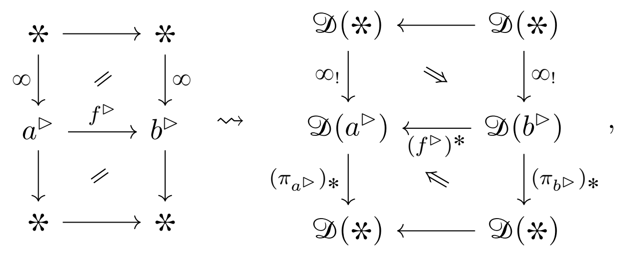
Lemma 1.3. *The assignment $a \mapsto \omega_a$ can be made into a functor*

$$\omega \colon \mathbf{Fin}^{\mathrm{op}} \to \mathrm{END}_{\mathbf{CAT}}(\mathscr{D}(*)),$$

i. e. for each $a \in \mathrm{ob}\,\mathbf{Fin}$, ω_a is an endofunctor of $\mathscr{D}(*)$ and we can assign to each map $f \colon a \to b$ between finite sets a natural transformation $\omega_f \colon \omega_b \Rightarrow \omega_a$ s. t. this assignment is compatible with compositions and identities.

Proof. For $a \in \mathrm{ob}\,\mathbf{Fin}$, ω_a is an endofunctor of $\mathscr{D}(*)$ by construction.

For functoriality, we consider $a, b \in \mathrm{ob}\,\mathbf{Fin}$ and $f \colon a \to b$. Then we have two diagrams



(1)

where the second one is obtained from the first by applying $\mathscr{D}$ and then using the appropriate mates.

Now we want to show that the upper natural transformation on the right is an isomorphism and then define the natural transformation $\omega_f \colon \omega_b \Rightarrow \omega_a$ as the pasting of the two squares on the right.

Note that we can detect such isomorphisms pointwise. In order to do that, we consider an $x \in \operatorname{ob}(a^{\triangleright})$, which yields a diagram

$$\begin{array}{ccccc} (\infty/x) & \xrightarrow{\pi} & \ast & \longrightarrow & \ast \\ {\scriptstyle\pi}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle\infty} & /\!/ & \big\downarrow{\scriptstyle\infty} \\ \ast & \xrightarrow[x]{} & a^{\triangleright} & \xrightarrow[f^{\triangleright}]{} & b^{\triangleright} \end{array} .$$

Then we know that the mate transformation $\pi_! \pi^* \Rightarrow x^* \infty_!$ is an isomorphism since the square on the left is a slice square and hence homotopy exact. Furthermore, we have

$$(\infty/x) \cong \begin{cases} \varnothing & x \neq \infty \\ \ast & x = \infty \end{cases} .$$

Since $f^{\triangleright}(x) = \infty$ iff $x = \infty$, this yields that the pasting of the two squares is also a slice square, hence homotopy exact, which means that the mate transformation $\pi_! \pi^* \Rightarrow (f^{\triangleright}(x))^* \infty_!$ is an isomorphism. Hence, in total, we obtain that the mate transformation $x^* \infty_! \Rightarrow (f^{\triangleright}(x))^* \infty_!$ is an isomorphism.

We can now define $\omega_f \colon \omega_b \Rightarrow \omega_a$ to be the pasting of the inverse of $\infty_! \Rightarrow (f^{\triangleright})^* \infty_!$ with $(\pi_{b^{\triangleright}})_* \Rightarrow (\pi_{a^{\triangleright}})_* (f^{\triangleright})^*$. This construction is compatible with composition of maps since mates are compatible with pastings. Furthermore, identities are mapped to identities since all the natural transformations in (1) are identities if f is an identity map. (See Remark 0.11 for an elaboration of these facts.) □

We will obtain a simplicial object from $\omega_{(\_)}$ by considering the simplex category as a subcategory of **Fin** in the following sense.

Remark 1.4. Let $\mathbf{\Delta}$ be the simplex category (cf. Notation A.1). Then we have a functor $\iota \colon \mathbf{\Delta} \to \mathbf{Fin}$ given by $\iota([n]) = \langle n \rangle$ on objects and $\iota(f) = f$ (as a map between sets) on morphisms. Note that ι is injective on objects and faithful.

Corollary 1.5. *Let $X \in \operatorname{ob}(\mathscr{D}(\ast))$. Then $(\omega_n X)_{n \in \mathbb{N}}$ together with the morphisms given by $(\omega_{\iota(f)})_X$ for $f \in \operatorname{mor} \mathbf{\Delta}$ is a simplicial object in $\mathscr{D}(\ast)$.*

Loop Objects as Monoid Objects

Our next step is showing that the simplicial objects associated with loop objects are *special* (cf. Definition A.5), i. e. are trivial in the zeroth level and satisfy the Segal condition. This will directly imply that loop objects have a monoid object structure.

Remark 1.6. For $X \in \operatorname{ob}(\mathscr{D}(\ast))$ we have $\omega_0 X \cong (\pi_{\lrcorner_0})_* \infty_! X \cong (\infty_! X)_0$ since 0 is the initial object of $\lrcorner_0$ (cf. Remark 0.19). Hence we obtain $\omega_0 X \cong (\infty_! X)_0 \cong 0$ since $\infty \colon \ast \to \langle 0 \rangle$ is a cosieve and $0 \notin \infty(\operatorname{ob} \ast)$ (cf. Remark 0.18).

This means a posteriori that the unique morphism $\omega_0 X \to 0$ is an isomorphism since the unique morphism between two zero objects is always an isomorphism.

Now we want to deal with higher levels.

Proposition 1.7. *Let $n > 1$. We define $i_n \colon \langle n-1 \rangle \to \langle n \rangle$ to be the inclusion and $i'_n \colon \langle 1 \rangle \to \langle n \rangle$ to be the function with $i'_n(0) = n-1$ resp. $i'_n(1) = n$.*

Then the natural transformation $\alpha_n \colon \omega_n \Rightarrow \omega_{n-1} \times \omega_1$ induced by the functor $k_n := i_n^{\triangleright} \amalg i'^{\,\triangleright}_n \colon \lrcorner_{n-1} \amalg \lrcorner_1 \to \lrcorner_n$ is an isomorphism.

Proof. Let J_n be the category which is obtained from $\lrcorner_n$ by adding two objects w_0, w_1 with morphisms $w_0 \to k$ for $0 \leqslant k \leqslant n-1$ resp. $w_1 \to k$ for $n-1 \leqslant k \leqslant n$ and resulting compositions such that all compositions to ∞ are equal. Let $j_n \colon \lrcorner_n \to J_n$ denote its inclusion functor.

Let $\lrcorner$ be the full subcategory of J_n containing w_0, w_1 and $n-1$ (which is isomorphic to $\lrcorner_1$), and let l_n denote its inclusion functor. Since $n-1$ is terminal in $\lrcorner$, we will denote it also by ∞. Note that l_n has a right adjoint r_n given by

$$r_n(x) = \begin{cases} w_0 & x \in \{w_0, 0, \ldots, n-2\} \\ w_1 & x \in \{w_1, n\} \\ n-1 & x \in \{n-1, \infty\} \end{cases} .$$

for $x \in \operatorname{ob} J_n$, which defines the images of morphisms uniquely. Hence we have $l_n^* \cong (r_n)_*$.

Then, using the natural equivalence $\mathscr{D}(A \amalg B) \simeq \mathscr{D}(A) \times \mathscr{D}(B)$ for $A, B \in \operatorname{ob} \mathbf{Cat}$ and appropriate mates, we obtain a diagram

$$\begin{array}{ccccc}
\mathscr{D}(\ast) & \xrightarrow{(\mathrm{id} \amalg \mathrm{id})^*} & \mathscr{D}(\ast \amalg \ast) & & \\
{\scriptstyle (\infty_n)_!}\downarrow & \Swarrow & \downarrow{\scriptstyle (\infty_{n-1} \amalg \infty_1)_!} & & \\
\mathscr{D}(\lrcorner_n) & \xrightarrow{k_n^*} & \mathscr{D}(\lrcorner_{n-1} \amalg \lrcorner_1) & & \\
{\scriptstyle (j_n)_*}\downarrow & \Nearrow & \downarrow{\scriptstyle (\pi_{n-1} \amalg \pi_1)_*} & & \\
\mathscr{D}(J_n) & \xrightarrow{(w_0 \amalg w_1)^*} & \mathscr{D}(\ast \amalg \ast) & \xrightarrow{(\pi_{\ast \amalg \ast})_*} & \mathscr{D}(\ast) \\
{\scriptstyle l_n^* \cong (r_n)_*}\downarrow & \Nearrow & \downarrow{\scriptstyle (w_0 \amalg w_1)_*} \; \approx & & \downarrow{\scriptstyle \mathrm{id}_*} \\
\mathscr{D}(\lrcorner) & \xrightarrow{\mathrm{id}^*} & \mathscr{D}(\lrcorner) & \xrightarrow{(\pi_1)_*} & \mathscr{D}(\ast)
\end{array} .$$

Under the equivalences mentioned above the upper natural transformation is given by

$$(\infty_{n-1})_! \times (\infty_1)_! \Rightarrow \left((i_n^{\triangleright})^* \times (i'^{\,\triangleright}_n)^* \right) (\infty_n)_!,$$

which is the product of the natural transformations which occur in the definition of ω_{i_n} resp. $\omega_{i'_n}$ (cf. (1) in the proof of Lemma 1.3). Hence it is an isomorphism as product of two natural isomorphisms. Furthermore, the right square in the last row commutes up to isomorphism since $\pi_1(w_0 \amalg w_1) = \pi_{\circledast\amalg\circledast}$.

All in all, the diagram above yields a natural transformation from

$$(\pi_1)_*(r_n)_*(j_n)_*(\infty_n)_! \cong (\pi_n)_*(\infty_n)_! \cong \omega_n$$

to

$$(\pi_{\circledast\amalg\circledast})_*(\pi_{n-1}\amalg\pi_1)_*(\infty_{n-1}\amalg\infty_1)_! \cong ((\pi_{n-1})_*(\infty_{n-1})_!)\times((\pi_1)_*(\infty_1)_!) \cong \omega_{n-1}\times\omega_1$$

which is the α_n mentioned in the statement of this proposition. We now want to show that (certain restrictions of) the natural transformations in the remaining two squares are isomorphisms, which will imply that α_n is an isomorphism.

For the middle square we consider diagrams of the form

$$\begin{array}{ccccc} (x/(\pi_{n-1}\amalg\pi_1)) & \xrightarrow{p_{x,\pi_{n-1}\amalg\pi_1}} & \lrcorner_{n-1}\amalg\lrcorner_1 & \xrightarrow{k_n} & \lrcorner_n \\ \downarrow{\scriptstyle \pi_{(x/(\pi_{n-1}\amalg\pi_1))}} & \Nearrow & \downarrow{\scriptstyle \pi_{n-1}\amalg\pi_1} & \Nearrow & \downarrow{\scriptstyle j_n} \\ \circledast & \xrightarrow[x]{} & \circledast\amalg\circledast & \xrightarrow[w_0\amalg w_1]{} & J_n \end{array}$$

for $x \in \mathrm{ob}(\circledast \amalg \circledast) = \{*_0, *_1\}$.

Then we have

$$(*_0/(\pi_{n-1}\amalg\pi_1)) \cong \lrcorner_{n-1} \quad \text{and} \quad (*_1/(\pi_{n-1}\amalg\pi_1)) \cong \lrcorner_1$$

where under this identification $p_{*_0,\pi_{n-1}\amalg\pi_1}$ resp. $p_{*_1,\pi_{n-1}\amalg\pi_1}$ is given by the inclusion ι_0 resp. ι_1 of the corresponding category. Since the left square is a slice square, this means that

$$*_0^*(\pi_{n-1}\amalg\pi_1)_* \Rightarrow (\pi_{(*_0/(\pi_{n-1}\amalg\pi_1))})_*(p_{*_0,\pi_{n-1}\amalg\pi_1})^* \cong (\pi_{n-1})_*\iota_0^*$$

and

$$*_1^*(\pi_{n-1}\amalg\pi_1)_* \Rightarrow (\pi_{(*_1/(\pi_{n-1}\amalg\pi_1))})_*(p_{*_1,\pi_{n-1}\amalg\pi_1})^* \cong (\pi_1)_*\iota_1^*$$

are isomorphisms.

On the other hand, we also have

$$(w_0/j_n) \cong \lrcorner_{n-1} \quad \text{and} \quad (w_1/j_n) \cong \lrcorner_1,$$

where under this identification p_{w_0,j_n} is given by $i_n^{\triangleright} = k_n\iota_0$ and p_{w_1,j_n} is given by $i_n'^{\triangleright} = k_n\iota_1$. Hence the pasting of the above squares is (up to isomorphisms) also a slice square, so the natural transformations

$$((w_0\amalg w_1)*_0)^*(j_n)_* = w_0^*(j_n)_* \Rightarrow (\pi_{(w_0/j_n)})_*(p_{w_0,j_n})^* \cong (\pi_{n-1})_*(i_n^{\triangleright})^*$$

and

$$((w_0 \amalg w_1)*_1)^*(j_n)_* = w_1^*(j_n)_* \Rightarrow (\pi_{(w_1/j_n)})_*(p_{w_1,j_n})^* \cong (\pi_1)_*(i_n^{\triangleright})^*$$

are also isomorphisms.

Combining these isomorphisms, we see that

$$\begin{aligned} *_0^*(\pi_{n-1} \amalg \pi_1)_* k_n^* &\cong (\pi_{n-1})_* \iota_0^* k_n^* \\ &\cong (\pi_{n-1})_*(i_n^{\triangleright})^* \cong w_0^*(j_n)_* \\ &\cong ((w_0 \amalg w_1)*_0)^*(j_n)_* \cong *_0^*(w_0 \amalg w_1)^*(j_n)_* \end{aligned}$$

and

$$\begin{aligned} *_1^*(\pi_{n-1} \amalg \pi_1)_* k_n^* &\cong (\pi_{n-1})_* \iota_1^* k_n^* \\ &\cong (\pi_{n-1})_*(i_n'^{\triangleright})^* \cong w_1^*(j_n)_* \\ &\cong ((w_0 \amalg w_1)*_1)^*(j_n)_* \cong *_1^*(w_0 \amalg w_1)^*(j_n)_*. \end{aligned}$$

Since mates are compatible with pastings this means that the natural transformation $x^*(w_0 \amalg w_1)^*(j_n)_* \Rightarrow x^*(\pi_{n-1} \amalg \pi_1)_* k_n^*$ is an isomorphism for all $x \in \mathrm{ob}(\star \amalg \star)$, hence it is an isomorphism as isomorphisms can be detected pointwise.

Note that, in general, the natural transformation

$$l_n^* \cong (r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$$

in the first square of the last row is not an isomorphism for all $X \in \mathrm{ob}(\mathscr{D}(J_n))$. We are going to "fix" this by restricting our attention to $\mathrm{essim}((j_n)_*(\infty_n)_!)$.

First, we compute $(n-1)^*(j_n)_* X'$ for $X' \in \mathrm{essim}(\infty_!)$: Consider the slice square

$$\begin{array}{ccc} (n-1/j_n) & \xrightarrow{p} & \lrcorner_n \\ {\scriptstyle \pi}\downarrow & \nearrow & \downarrow{\scriptstyle j_n} \\ \star & \xrightarrow[n-1]{} & J_n \end{array} .$$

Then we know that $(n-1)^*(j_n)_* \Rightarrow \pi_* p^*$ is an isomorphism.

Now $(n-1/j_n)$ is isomorphic to the full subcategory K_n of $\lrcorner_n$ spanned by $n-1$ and ∞, where p corresponds to the inclusion $K_n \to \lrcorner_n$ under this identification. Hence we see that $\pi_* p^* \cong (n-1)^* p^* \cong (p(n-1))^*$ since $n-1$ is the initial object of K_n. Now note that $p(n-1)$ is the inclusion of $n-1$ into $\lrcorner_n$. Therefore $(n-1)^*(j_n)_* \cong (n-1)^*$, where the former $n-1$ is the object in J_n and the latter the one in $\lrcorner_n$. Since $\infty\colon \star \to \lrcorner_n$ is a cosieve we know that $(n-1)^* X' \cong 0$ for $X' \in \mathrm{essim}(\infty_!)$, so we obtain $(n-1)^*(j_n)_* X' \cong 0$.

This means that for $X \in \mathrm{essim}((j_n)_*(\infty_n)_!)$ we have $\infty^* l_n^* X \cong (l_n \infty)^* X \cong (n-1)^* X \cong 0$. On the other hand, for any $Y \in \mathscr{D}(\star \amalg \star)$, we have

$\infty^*(w_0 \amalg w_1)_* Y \cong 0$ since $w_0 \amalg w_1$ is a sieve. Hence $l_n^* \cong (r_n)_*$ and $(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ agree on $\infty = n-1$ for $X \in \operatorname{essim}((j_n)_*(\infty_n)_!)$.

We now consider w_i for $i \in \{0,1\}$. In the slice square

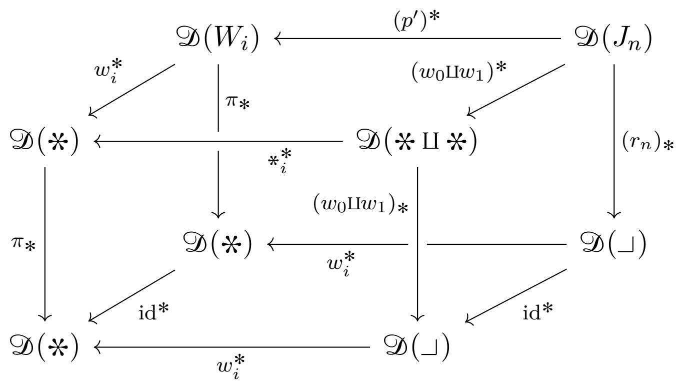
$$\begin{array}{ccc} (w_i/w_0 \amalg w_1) & \xrightarrow{\;p\;} & \ast \amalg \ast \\ {\scriptstyle\pi}\downarrow & \nearrow & \downarrow{\scriptstyle w_0 \amalg w_1} \\ \ast & \xrightarrow[w_i]{} & \lrcorner \end{array},$$

$(w_i/w_0 \amalg w_1)$ can be identified with $\ast$ and p with $*_i \colon \ast \to \ast \amalg \ast$. On the other hand, in the slice square

$$\begin{array}{ccc} (w_i/r_n) & \xrightarrow{\;p'\;} & J_n \\ {\scriptstyle\pi}\downarrow & \nearrow & \downarrow{\scriptstyle r_n} \\ \ast & \xrightarrow[w_i]{} & \lrcorner \end{array},$$

(w_i/r_n) can be identified with the subcategory W_i of J_n spanned by objects under w_i, i. e. objects x s. t. there exists a morphism $f\colon w_i \to x$. Under this identification p' becomes the inclusion of W_i.

Hence we obtain a cube



,

where the faces are filled with the natural transformations considered above.

Now upper and lower faces commute by the functoriality of $\mathscr{D}$. Hence the pasting of the back face with the left face coincides with the pasting of the right face with the front face since both are induced by the equality of $r_n p' w_i = r_n(w_0 \amalg w_1) *_i$ to w_i.

The left face commutes up to isomorphism since w_i is the initial object of W_i and hence $(\pi_{W_i})_* \cong w_i^*$ holds. Moreover, the front face and the back face are induced by slice squares, so they are also filled with isomorphisms. Hence the pasting of $(r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ with $w_i^*(w_0 \amalg w_1)_* \Rightarrow \pi_* *_i^*$ is an isomorphism. Inverting $w_i^*(w_0 \amalg w_1)_* \Rightarrow \pi_* *_i^*$, we see that $w_i^*(r_n)_* \Rightarrow w_i^*(w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ is an isomorphism.

All in all, for $X \in \mathrm{essim}((j_n)_*(\infty_n)_!)$, $(r_n)_* \Rightarrow (w_0 \amalg w_1)_*(w_0 \amalg w_1)^*$ is an isomorphism pointwise, so it is indeed an isomorphism. This means that the last remaining natural transformation is also an isomorphism in the relevant case, so $\alpha_n \colon \omega_n \Rightarrow \omega_{n-1} \times w_1$ is an isomorphism in total. □

Thus we obtain a monoid object structure on $\omega_1 X \cong \Omega X$ for $X \in \mathrm{ob}(\mathscr{D}(\ast))$ as follows:

Corollary 1.8. *Let $\mu\colon \langle 1\rangle \to \langle 2\rangle$ be the map with $\mu(0) = 0$ resp. $\mu(1) = 2$ and let ε be the unique map from $\langle 1\rangle$ to $\langle 0\rangle$.*

Then for any $X \in \mathrm{ob}(\mathscr{D}(\ast))$, $\omega_1 X \cong \Omega X$ has a monoid object structure given by the multiplication

$$m_X \colon \omega_1 X \times \omega_1 X \xrightarrow{(\alpha_2^{-1})_X} \omega_2 X \xrightarrow{(\omega_\mu)_X} \omega_1 X$$

and the unit

$$0\colon 0 \xrightarrow{\cong} \omega_0 X \xrightarrow{(\omega_\varepsilon)_X} \omega_1 X.$$

Proof. The previous proposition and the preceding remark imply that the Segal morphism

$$\omega_n X \to (\omega_1 X)^n$$

is an isomorphism for any $n \in \mathbb{N}$. Then we have $\omega_0 X \cong 0$ and the simplicial object induced by $\omega_{(\_)} X \colon \mathbf{Fin}^{\mathrm{op}} \to \mathscr{D}(\ast)$ satisfies the Segal condition, so it is a special simplicial object. Therefore $\omega_1 X = \Omega X$ has a natural monoid object structure (cf. Proposition A.6).

Now note that $[n] \in \mathrm{ob}\,\mathbf{\Delta}$ and $\iota([n]) = \langle n\rangle \in \mathrm{ob}\,\mathbf{Fin}$ are equal as sets for all $n \in \mathbb{N}$. Furthermore, we have $i_2 = \phi^0$, $i_2' = \phi^1$, $\mu = \delta^1$ and $\varepsilon = \sigma^0$ as maps between sets (cf. Notation A.1). Hence the monoid object structure on $\omega_1 X$ which is induced by the special simplicial object corresponding to $\omega_{(\_)} X$ is indeed given by the morphisms m_X and 0. □

Loop Objects as Group Objects

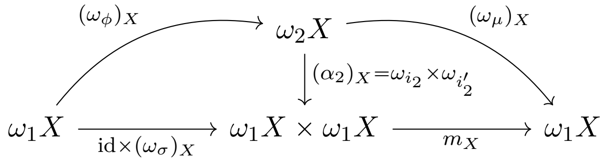
The last step in this section is the construction of inverses for the multiplication of loop objects, concluding that loop objects have a group object structure. The crucial point here is the additional structure **Fin** carries in comparison to $\mathbf{\Delta}$.

Proposition 1.9. *Let $\sigma\colon \langle 1\rangle \to \langle 1\rangle$ be the only non-trivial automorphism, i. e. the map swapping* 0 *and* 1*.*

Then, for any $X \in \mathrm{ob}(\mathscr{D}(\ast))$, the morphism given by $(\omega_\sigma)_X \colon \omega_1 X \to \omega_1 X$ is an inversion morphism for the multiplication m_X of $\Omega X \cong \omega_1 X$.

Proof. We have to show that the composition $z := m_X \circ (\mathrm{id}_X \times (w_\sigma)_X)$ factors through $\omega_0 X \cong 0$, i. e. is the zero morphism. In order to do this we will describe z as a morphism which factors through $\omega_2 X$.

Let $\phi\colon \langle 2\rangle \to \langle 1\rangle$ be the map with $\phi(0) = 0 = \phi(2)$ and $\phi(1) = 1$. We claim that the diagram



commutes.

The right triangle commutes by the definition of m_X. We verify the commutativity of the left triangle componentwise. Indeed, we have

$$\begin{aligned}\mathrm{pr}_1 \circ (\omega_{i_2} \times \omega_{i'_2}) \circ (\omega_\phi)_X &= \omega_{i_2} \circ (\omega_\phi)_X = (\omega_{\phi\circ i_2})_X = (\omega_{\mathrm{id}_{\langle 1\rangle}})_X \\ &= \mathrm{id}_{\omega_1 X} = \mathrm{pr}_1 \circ (\mathrm{id}_{\omega_1 X} \times (\omega_\sigma)_X)\end{aligned}$$

and

$$\begin{aligned}\mathrm{pr}_2 \circ (\omega_{i_2} \times \omega_{i'_2}) \circ (\omega_\phi)_X &= \omega_{i'_2} \circ (\omega_\phi)_X = (\omega_{\phi\circ i'_2})_X = (\omega_\sigma)_X \\ &= \mathrm{pr}_2 \circ (\mathrm{id}_{\omega_1 X} \times (\omega_\sigma)_X)\end{aligned}$$

since $\phi \circ i_2 = \mathrm{id}_{\langle 1\rangle}$ and $\phi \circ i'_2 = \sigma$.

Hence we obtain that

$$\begin{aligned}z &= m_X \circ (\mathrm{id}_X \times (w_\sigma)_X) \\ &= (\omega_\mu)_X \circ (\alpha_2^{-1})_X \circ (\alpha_2)_X \circ (\omega_\phi)_X \\ &= (\omega_\mu)_X \circ (\omega_\phi)_X = (\omega_{\phi\circ\mu})_X.\end{aligned}$$

Now note that $\phi \circ \mu$ factors through $\langle 0\rangle$ as $\phi(\mu(0)) = \phi(0) = 0 = \phi(2) = \phi(\mu(0))$. Hence $z = (\omega_{\phi\circ\mu})_X$ factors through $\omega_0 X \cong 0$. □

2 Double Loop Objects

The considerations of the previous section yield a group object structure on a twofold loop object $\Omega^2 X$ because it is the loop object of the object ΩX in $\mathscr{D}(\ast)$. In this section, we will show that this group object structure is abelian. In order to do that, we will first define a "new" group object structure on $\Omega^2 X$. Then, using a formal Eckmann–Hilton argument (cf. [2]), we will prove that these two structures coincide and are abelian.

In this section $\mathscr{D}$ will again be a pointed derivator.

Loop Functor as a Functor to Group Objects

An important result which we need is the fact that the loop functor factors through the category $\mathscr{D}(\ast)$-**Grp** of group objects in $\mathscr{D}(\ast)$ also on the level of morphisms.

Lemma 2.1. *Let $f\colon X \to Y$ be a morphism in $\mathscr{D}(\ast)$.*

Then the induced morphism $\Omega f\colon \Omega X \to \Omega Y$ is a homomorphism of group objects in $\mathscr{D}(\ast)$, where ΩX and ΩY are endowed with the group object structure discussed in the previous section.

Proof. First we note that the functors $\omega_{(\_)}X, \omega_{(\_)}Y\colon \mathbf{Fin}^{\mathrm{op}} \to \mathscr{D}(\ast)$ induce special simplicial objects as discussed in the previous section. Furthermore, a morphism $f\colon X \to Y$ induces morphisms $\omega_a f\colon \omega_a X \to \omega_a Y$ for $a \in \operatorname{ob}\mathbf{Fin}$. This assignment is natural in a since for a given $u\colon a \to b$, the diagram

$$\begin{array}{ccc} \omega_a X & \xrightarrow{\omega_a f} & \omega_a Y \\ {\scriptstyle (\omega_u)_X}\uparrow & & \uparrow{\scriptstyle (\omega_u)_Y} \\ \omega_b X & \xrightarrow[\omega_b f]{} & \omega_b Y \end{array}$$

commutes since ω_u is a natural transformation by Lemma 1.3.

Hence $\omega_{(\_)}f\colon \omega_{(\_)}X \Rightarrow \omega_{(\_)}Y$ induces a morphism of monoid objects

$$\Omega X = \omega_1 X \xrightarrow{\omega_1 f = \Omega f} \omega_1 Y = \Omega Y$$

as a natural transformation between special simplicial objects (cf. Proposition A.8). Now any morphism of monoid objects between group objects is already a morphism of group objects. (This can be, for example, checked on represented functors and hence can be reduced to the fact that a monoid homomorphism between groups is already a group homomorphism.) ☐

Products under the Loop Functor

Now we will prove another result that brings us closer to the Eckmann–Hilton argument, namely show that Ω preserves products.

Remark 2.2. Note that the functor $\Omega\colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ has a left adjoint $\Sigma\colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ (cf. [5, Proposition 8.18]).

Hence Ω preserves limits. In particular, the natural morphism

$$\Omega\left(\prod_{i\in I} X_i\right) \xrightarrow{\prod_{i\in I}\Omega(\mathrm{pr}_i)} \prod_{i\in I}\Omega X_i$$

is an isomorphism for any index set I and any family $(X_i)_{i\in I}$ of objects in $\mathscr{D}(\ast)$.

This immediately implies that also the group object structure on loop objects are compatible with products.

Remark 2.3. For $X, Y \in \operatorname{ob}\mathscr{D}(\ast)$, the isomorphism

$$\Omega(X \times Y) \xrightarrow{\Omega(\mathrm{pr}_1)\times\Omega(\mathrm{pr}_2)} \Omega X \times \Omega Y$$

is also a homomorphism of group objects since it is the product of two group object homomorphisms. Hence it is already an isomorphism of group objects since commutativity of compatibility diagrams for $\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2)$ implies the commutativity of compatibility diagrams for its inverse.

This endows $\Omega(X \times Y)$ with the structure of a product of ΩX and ΩY as group objects s. t.

$$\begin{aligned}\mathrm{mult}_{\Omega X\times\Omega Y} =&((\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2)) \times (\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2))) \circ m_{\Omega(X\times Y)}\\ &\circ (((\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2))^{-1} \times (\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2))^{-1}),\end{aligned}$$

where $\mathrm{mult}_{\Omega X\times\Omega Y}\colon (\Omega X \times \Omega Y) \times (\Omega X \times \Omega Y) \to \Omega X \times \Omega Y$ is the multiplication morphism of the product group object.

Furthermore, the compatibility of Ω with products yields a "new" group object structure on double loop objects.

Corollary 2.4. *For $X \in \operatorname{ob}(\mathscr{D}(\ast))$, $\Omega^2(X)$ has (in addition to the one given by being the loop object of $\Omega(X)$) a group object structure given by the multiplication*

$$m'_X\colon \Omega^2(X)\times\Omega^2(X) \xrightarrow{(\Omega(\mathrm{pr}_1)\times\Omega(\mathrm{pr}_2))^{-1}} \Omega(\Omega X\times\Omega X) \xrightarrow{\Omega(m_X)} \Omega(\Omega(X)) = \Omega^2 X,$$

the unit

$$0 \to \Omega^2 X$$

and inverses

$$\Omega^2 X \xrightarrow{\Omega((\omega_\sigma)_X)} \Omega^2 X.$$

Proof. The commutativity of the required diagrams follow from the fact that the corresponding diagrams commute before applying Ω. □

The Eckmann–Hilton Argument

We now have everything at hand to imitate the standard proof of the fact that a group object in **Grp** is an abelian group in order to show the main result of this section.

Lemma 2.5. *Let $X \in \operatorname{ob}\mathscr{D}(\ast)$. Let $s_{2,3} := \mathrm{pr}_1 \times \mathrm{pr}_3 \times \mathrm{pr}_2 \times \mathrm{pr}_4 \colon (\Omega^2 X)^4 \to (\Omega^2 X)^4$ be the morphism which "swaps the second and the third factor".*

Then the diagram

$$\begin{tikzcd}
\Omega^2X \times \Omega^2X \times \Omega^2X \times \Omega^2X \arrow[rr, "s_{2,3}"] \arrow[d, "m'_X \times m'_X"'] & & \Omega^2X \times \Omega^2X \times \Omega^2X \times \Omega^2X \arrow[d, "m_{\Omega X} \times m_{\Omega X}"] \\
\Omega^2X \times \Omega^2X \arrow[dr, "m_{\Omega X}"'] & & \Omega^2X \times \Omega^2X \arrow[dl, "m'_X"] \\
& \Omega^2 X &
\end{tikzcd}$$

is commutative.

Proof. We first note that the diagram

$$\begin{tikzcd}
& \Omega(\Omega X \times \Omega X) \times \Omega(\Omega X \times \Omega X) \arrow[dl, "\Omega(m_X) \times \Omega(m_X)"'] \arrow[dr, "m_{\Omega X \times \Omega X}"] & \\
\Omega(\Omega X) \times \Omega(\Omega X) \arrow[dr, "m_{\Omega X}"'] & & \Omega(\Omega X \times \Omega X) \arrow[dl, "\Omega(m_X)"] \\
& \Omega(\Omega X)) &
\end{tikzcd} \tag{2}$$

commutes since $\Omega(m_X) \colon \Omega(\Omega X \times \Omega X) \to \Omega(\Omega(X))$ is a homomorphism of group objects by Lemma 2.1.

Now $(m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3}$ is the multiplication morphism of $\Omega^2 X \times \Omega^2 X$. By Remark 2.3 this morphism also coincides with $m_{\Omega X \times \Omega X}$ up to "conjugation" with $(\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2)) \times (\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2))$.

Hence, identifying $\Omega(\Omega X \times \Omega X)$ with $\Omega^2 X \times \Omega^2 X$ via $\Omega(\mathrm{pr}_1) \times \Omega(\mathrm{pr}_2)$, the diagram (2) becomes a commutative diagram

$$\begin{tikzcd}
& \Omega^2X \times \Omega^2X \times \Omega^2X \times \Omega^2X \arrow[ddl, bend right, "m'_X \times m'_X"'] \arrow[dr, "(m_{\Omega X} \times m_{\Omega X}) \circ s_{2,3}"] & \\
& \Omega(\Omega X \times \Omega X) \times \Omega(\Omega X \times \Omega X) \arrow[u, "\cong"] \arrow[dl, "\Omega(m_X) \times \Omega(m_X)"] \arrow[dr, "m_{\Omega X \times \Omega X}"'] & \Omega^2X \times \Omega^2X \arrow[ddl, bend right, "m'_X"'] \\
\Omega(\Omega X) \times \Omega(\Omega X) \arrow[dr, "m_{\Omega X}"'] & & \Omega(\Omega X \times \Omega X) \arrow[u, "\cong"'] \arrow[dl, "\Omega(m_X)"] \\
& \Omega(\Omega X)) &
\end{tikzcd},$$

which contains the required diagram as its front square. □

Corollary 2.6. *The "group laws"* $m_{\Omega X}$ *and* m'_X *on* $\Omega^2 X$ *coincide and are abelian.*

In particular, $\Omega^2 \colon \mathscr{D}(\ast) \to \mathscr{D}(\ast)$ *factors through the category* $\mathscr{D}(\ast)$-**Ab** *of abelian group objects in* $\mathscr{D}(\ast)$.

Proof. Consider the morphism

$$f \coloneqq \mathrm{pr}_1 \times 0 \times 0 \times \mathrm{pr}_2 \colon \Omega^2 X \times \Omega^2 X \to \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X.$$

Then we have $\mathrm{pr}_1 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f = m_{\Omega_X} \circ (\mathrm{pr}_1 \times 0) = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f = m_{\Omega_X} \circ (\mathrm{pr}_2 \times 0) = \mathrm{pr}_1$ since $0 \to \Omega^2 X$ is the unit morphism for m_{Ω_X}. Hence we have $(m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f = \mathrm{id}_{\Omega^2 X \times \Omega^2 X}$ as these morphisms agree after composing with each of the projections.

Furthermore, we also have $\mathrm{pr}_1 \circ (m'_X \times m'_X) \circ f = m'_X \circ (\mathrm{pr}_1 \times 0) = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (m'_X \times m'_X) \circ f = m'_X \circ (\mathrm{pr}_2 \times 0) = \mathrm{pr}_1$ since $0 \to \Omega^2 X$ is also the unit morphism for m'_X. Hence $(m'_X \times m'_X) \circ f = \mathrm{id}_{\Omega^2 X \times \Omega^2 X}$ as these agree after composing with each of the projections.

In total, using the Eckmann–Hilton identity from the previous lemma, we obtain

$$\begin{aligned} m_{\Omega X} &= m_{\Omega X} \circ \mathrm{id}_{\Omega^2 X \times \Omega^2 X} \\ &= m_{\Omega X} \circ (m'_X \times m'_X) \circ f \\ &= m'_X \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ f \\ &= m'_X \circ \mathrm{id}_{\Omega^2 X \times \Omega^2 X} = m'_X. \end{aligned}$$

For the commutativity of $m_{\Omega X} = m'_X$ we consider the morphism

$$g \coloneqq 0 \times \mathrm{pr}_1 \times \mathrm{pr}_2 \times 0 \colon \Omega^2 X \times \Omega^2 X \to \Omega^2 X \times \Omega^2 X \times \Omega^2 X \times \Omega^2 X.$$

Then we have $\mathrm{pr}_1 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g = m_{\Omega_X} \circ (0 \times \mathrm{pr}_2) = \mathrm{pr}_2$ and $\mathrm{pr}_2 \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g = m_{\Omega_X} \circ (\mathrm{pr}_1 \times 0) = \mathrm{pr}_1$, therefore $(m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g = \mathrm{pr}_2 \times \mathrm{pr}_1$, i. e. the "swapping morphism". On the other hand, we also have $\mathrm{pr}_1 \circ (m'_X \times m'_X) \circ g = m'_X \circ (0 \times \mathrm{pr}_1) = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (m'_X \times m'_X) \circ g = m'_X \circ (\mathrm{pr}_2 \times 0) = \mathrm{pr}_2$, so $(m'_X \times m'_X) \circ g = \mathrm{id}_{\Omega^2 X \times \Omega^2 X}$.

Hence, the Eckmann–Hilton identity yields

$$\begin{aligned} m_{\Omega X} &= m_{\Omega X} \circ \mathrm{id}_{\Omega^2 X \times \Omega^2 X} \\ &= m_{\Omega X} \circ (m'_X \times m'_X) \circ g \\ &= m'_X \circ (m_{\Omega_X} \times m_{\Omega X}) \circ s_{2,3} \circ g \\ &= m'_X \circ (\mathrm{pr}_2 \times \mathrm{pr}_1) \\ &= m_{\Omega X} \circ (\mathrm{pr}_2 \times \mathrm{pr}_1), \end{aligned}$$

which means that $m_{\Omega X} = m'_X$ is indeed a commutative multiplication.

Now any homomorphism of group objects between abelian group objects is a homomorphism of abelian group objects and vice versa. Therefore the commutativity of $m_{\Omega X} = m'_X$ already implies that Ω^2 factors through $\mathscr{D}(*)$-**Ab**. $\square$

Lastly, let us observe that we did not need any specifics about derivators or the loop functor for most of the statements in this section. Indeed, proofs of all of the statements from Remark 2.3 to Corollary 2.6 can be done for any category $\mathscr{C}$ with finite products together with a functor $F\colon \mathscr{C} \to \mathscr{C}$ which factors through $\mathscr{C}$-**Grp** and preserves products in $\mathscr{C}$. In particular, in that case, F^2 factors through $\mathscr{C}$-**Ab**.

3 Applications

Besides the intrinsic motivation for studying it, the loop functor can be used to show that values of a certain type of derivators are additive.

Definition 3.1. A pointed derivator $\mathscr{D}$ is called *stable* if the loop functor $\Omega\colon \mathscr{D}(*) \to \mathscr{D}(*)$ is an equivalence of categories.

Typical examples of stable derivators include the homotopy derivator of chain complexes of modules over a ring and the homotopy derivator of spectra (cf. [5, Section 9]).

The statements we have proven about the loop functor imply the following result about stable derivators.

Corollary 3.2. *Let $\mathscr{D}$ be a stable derivator. Then $\mathscr{D}(*)$ is an additive category.*

Proof. First of all, we know that Ω (and hence Ω^2) is an equivalence of categories since $\mathscr{D}$ is stable. Now note that $\Omega^2\colon \mathscr{D}(*) \to \mathscr{D}(*)$ factors through the category $\mathscr{D}(*)$-**Ab** of abelian group objects in $\mathscr{D}(*)$ by Corollary 2.6 since morphisms between abelian group objects are simply homomorphisms of underlying group objects.

We know that $\mathrm{Hom}_{\mathscr{D}(*)\text{-}\mathbf{Ab}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{D}(*)}(A, B)$ for all $A, B \in \mathrm{ob}\,\mathscr{D}(*)\text{-}\mathbf{Ab}$. Now, for $X, Y \in \mathscr{D}(*)$, the map induced by Ω^2 on homomorphism sets is a bijection onto $\mathrm{Hom}_{\mathscr{D}(*)}\big(\Omega^2 X, \Omega^2 Y\big)$ which factors through $\mathrm{Hom}_{\mathscr{D}(*)\text{-}\mathbf{Ab}}\big(\Omega^2 X, \Omega^2 Y\big)$. Hence we have

$$\mathrm{Hom}_{\mathscr{D}(*)}\big(\Omega^2 X, \Omega^2 Y\big) = \mathrm{Hom}_{\mathscr{D}(*)\text{-}\mathbf{Ab}}\big(\Omega^2 X, \Omega^2 Y\big)\,,$$

and Ω^2 is fully faithful also as a functor to $\mathscr{D}(*)$-**Ab**.

This means that the essential image $\mathscr{A}$ of Ω^2 in $\mathscr{D}(*)$-**Ab** is equivalent to $\mathscr{D}(*)$. $\mathscr{A}$ has finite products since $\mathscr{D}(*)$ has finite products as the underlying category of a derivator (cf. Remark 0.16). Furthermore, $\mathscr{A}$ is enriched over **Ab** as a full subcategory of the additive category $\mathscr{D}(*)$-**Ab** (cf. Corollary B.10). Hence $\mathscr{A} \simeq \mathscr{D}(*)$ is an additive category by Corollary B.9. □

In fact, in the stable case $\mathscr{D}(*)$ is equivalent to $\mathscr{D}(*)$-**Ab** as it is the case for all additive categories.

Now we want to apply our results to all values $\mathscr{D}(A)$ of a derivator $\mathscr{D}$.

Definition 3.3. Let A be a small category and $\mathscr{D}$ a derivator.

Then the *shifted derivator* $\mathscr{D}^A$ is given by $\mathscr{D}^A(B) = \mathscr{D}(A \times B)$ on small categories, $\mathscr{D}^A(u) = \mathscr{D}(\mathrm{id}_A \times u)$ on functors and $\mathscr{D}^A(\gamma) = \mathscr{D}(\mathrm{id}_{\mathrm{id}_A} \times \gamma)$ on natural transformations (cf. [5, Proposition 7.32]).

$\mathscr{D}^A$ is pointed respectively stable if $\mathscr{D}$ is so, hence we can obtain statements about $\mathscr{D}(A) \simeq \mathscr{D}^A(*)$ by considering $\mathscr{D}^A$ as a derivator.

Remark 3.4. Let A be a small category and $\mathscr{D}$ a pointed derivator.

Then the shifted loop functor

$$\Omega^A := (\mathrm{id}_A \times \pi_\lrcorner)_* \circ (\mathrm{id}_A \times \infty)_! \colon \mathscr{D}(A) \to \mathscr{D}(A)$$

factors through $\mathscr{D}$(A)-**Grp** and the twofold shifted loop functor $\left(\Omega^A\right)^2$ factors even through $\mathscr{D}$(A)-**Ab**.

Moreover, $\mathscr{D}(A)$ is an additive category if $\mathscr{D}$ (and hence $\mathscr{D}^A$) is stable.

A The Segal Condition

In this appendix we will justify Corollary 1.8 by showing that a certain type of simplicial objects gives rise to monoid objects.

We start with a review of simplicial objects, a concept which generalizes simplicial sets. A more detailed introduction to simplicial sets can be found in [8, Chapter 3].

Notation A.1. Let $\mathbf{\Delta}$ be the simplex category, i. e. the category of non-empty finite ordinal numbers. For $n \in \mathbb{N}$ set $[n] = \{0, \dots, n\}$.

For $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$ we fix notation for the following morphisms in $\mathbf{\Delta}$:

- $\delta^{n,i}\colon [n-1] \to [n]$, $n > 0$, is the unique monomorphism which "skips i",
- $\sigma^{n,i}\colon [n+1] \to [n]$ is the unique epimorphism which "collapses $i+1$ to i",
- $\phi^{n,i}\colon [1] \to [n]$, $i < n$, is the inclusion of $\{i, i+1\}$.

In most cases, we will omit the index n if it is clear from context.

Given a category $\mathscr{C}$ and a simplicial object $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$, we will denote $X([n])$ by X_n. Then the above maps induce:

- $d_i^X := X(\delta^i)\colon X_n \to X_{n-1}$, the *i-th face morphism*,
- $s_i^X := X(\sigma^i)\colon X_n \to X_{n+1}$, the *i-th degeneracy morphism*,
- $f_i^X := X(\phi^i)\colon X_n \to X_1$.

The simplicial object in consideration will mostly be clear from context and we will omit the upper index X in these cases.

Now we have a closer look at relations between face and degeneracy morphisms.

Remark A.2. All morphisms in $\mathbf{\Delta}$ can be written as a composition of suitable δ^i's and σ^i's. These maps satisfy the *simplicial relations*:

- $\delta^j \circ \delta^i = \delta^i \circ \delta^{j-1}$ for $i < j$,
- $\sigma^j \circ \delta^i = \delta^i \circ \sigma^{j-1}$ for $i < j$,
- $\sigma^j \circ \delta^i = \mathrm{id}$ for $i = j$ and $i = j+1$,
- $\sigma^j \circ \delta^i = \delta^{i-1} \circ \sigma^j$ for $i > j$,
- $\sigma^j \circ \sigma^i = \sigma^{i-1} \circ \sigma^j$ for $i > j$.

Furthermore, all relations between the δ^i's and the σ^i's are implied by these relations in the following sense:

For a category $\mathscr{C}$, a collection $(X_n)_{n\in\mathbb{N}}$ of objects in $\mathscr{C}$ with morphisms $d_i\colon X_{n-1} \to X_n$ for $n > 0$, $0 \leqslant i \leqslant n$ and $s_i\colon X_n \to X_{n+1}$ for $0 \leqslant i \leqslant n$ yields a simplicial object X s. t. $d_i = X(\delta^i)$ and $s_i = X(\sigma^i)$ iff the *simplicial identities* (which are induced by the simplicial relations) hold:

- $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$,
- $d_i \circ s_j = s_{j-1} \circ d_i$ for $i < j$,
- $d_i \circ s_j = \mathrm{id}$ for $i = j$ and $i = j + 1$,
- $d_i \circ s_j = s_j \circ d_{i-1}$ for $i > j$,
- $s_i \circ s_j = s_j \circ s_{i-1}$ for $i > j$.

A relevant fact in the theory of simplicial sets is the following characterization of nerves of small categories which appeared in [10].

Remark A.3. Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set.

We define a family $(\tilde{X}_n)_{n\in\mathbb{N}}$ together with a maps $(g_n\colon \tilde{X}_n \to X_0)_{n\in\mathbb{N}}$ as follows: Let $\tilde{X}_0 := X_0$ and $g_0 = \mathrm{id}_{X_0}$. Now let $\tilde{X}_k$ and $g_k\colon \tilde{X}_k \to X_0$ be given. Then we define $\tilde{X}_{k+1}$ to be the fiber product $\tilde{X}_k \times_{X_0} X_1$, where the structure maps are given by g_k resp. d_1. Furthermore, we define g_{k+1} to be the map $d_0 \circ \mathrm{pr}_2\colon \tilde{X}_k \times_{X_0} X_1 \to X_0$.

Note that, for $n > 0$, the iterated fiber product $\tilde{X}_n$ can be seen as a subset of $(X_1)^n$ and $\prod_{i=0}^{n-1} f_i\colon X_n \to (X_1)^n$ factors through the inclusion of $\tilde{X}_n$.

Furthermore, a simplicial set $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to the nerve of a (small) category if and only if the *Segal condition* is satisfied, i. e. for any $n > 0$, the natural map

$$X_n \xrightarrow{\prod_{i=0}^{n-1} f_i} \tilde{X}_n,$$

is a bijection.

This immediately yields an alternative characterization of monoids.

Corollary A.4. *Simplicial sets $X \in \mathrm{ob}(\mathbf{sSet})$ which have exactly one 0-simplex (i. e. $X_0 \cong \{*\}$) and fulfill the Segal condition can be identified with monoids since small categories with only one object can be identified with monoids.*

In the following we want to prove a similar statement for simplicial objects in a category. For this, some generalities on monoid objects will be required, which can be found in [2] or [9, Section III.6].

We know that, in general, fiber products don't exist in values of a derivator, but products do (cf. Remark 0.16). Therefore we will restrict our attention to simplicial objects X with $X_0 \cong *$ for a terminal object $*$, so that fiber products over X_0 are usual products.

In the rest of this appendix $\mathscr{C}$ will be a category which has finite products (hence an terminal object $*$) and $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$ a simplicial object in $\mathscr{C}$.

Definition A.5. X is called *special* if $\pi_{X_0}\colon X_0 \to *$ is an isomorphism and X satisfies the Segal condition, i. e.

$$X_n \xrightarrow{\prod_{i=0}^{n-1} f_i} X_1^n$$

is an isomorphism for $n > 0$.

We will denote the category of special simplicial objects in $\mathscr{C}$ with natural transformations between those as morphisms by $(\mathbf{s}\mathscr{C})^{\mathrm{sp}}$.

First we show that special simplicial objects give rise to monoid objects.

Proposition A.6. *Let X be a special simplicial object in $\mathscr{C}$.*

Then X_1 has a monoid object structure given by the multiplication morphism

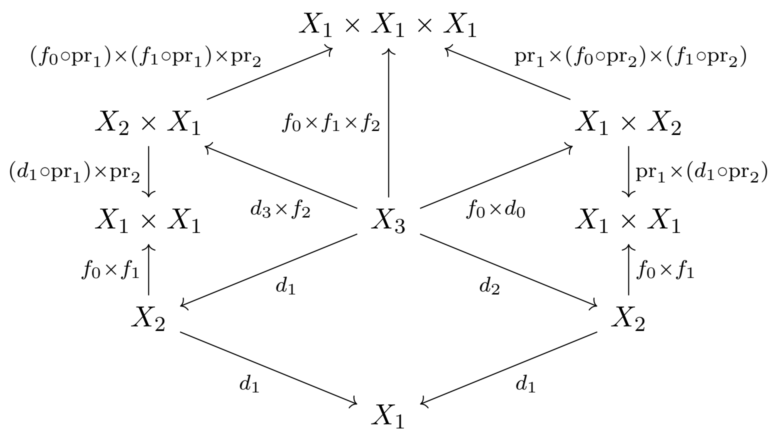
$$m_X\colon X_1 \times X_1 \xrightarrow{(f_0\times f_1)^{-1}} X_2 \xrightarrow{d_1} X_1$$

and the unit morphism

$$e_X\colon * \xrightarrow{(\pi_{X_0})^{-1}} X_0 \xrightarrow{s_0} X_1,$$

where we will omit the index X if it is clear from context.

Proof. For associativity we consider the diagram



.

The lower parallelogram commutes as $d_1 \circ d_2 = d_1 \circ d_1$ is one of the simplicial identities. Note that the upper left and upper right sides of the diagram are symmetric, so we will only show that the upper left part is commutative since the commutativity of the other part can be shown similarly.

For the upper left triangle we have

$$\begin{aligned}\mathrm{pr}_1 \circ ((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= (f_0 \circ \mathrm{pr}_1) \circ (d_3 \times f_2)\\ &= f_0 \circ d_3 = f_0\\ &= \mathrm{pr}_1 \circ (f_0 \times f_1 \times f_2)\end{aligned}$$

since $\phi^0 = \delta^3 \circ \phi^0 \colon [1] \to [2] \to [3]$. Similarly, we also have

$$\begin{aligned}\mathrm{pr}_2 \circ ((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= f_1 \circ d_3\\ &= f_1 = \mathrm{pr}_2 \circ (f_0 \times f_1 \times f_2)\end{aligned}$$

since $\phi^1 = \delta^3 \circ \phi^1 \colon [1] \to [2] \to [3]$. For the third factor we have

$$\begin{aligned}\mathrm{pr}_3 \circ ((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= \mathrm{pr}_2 \circ (d_3 \times f_2)\\ &= f_2 = \mathrm{pr}_3 \circ (f_0 \times f_1 \times f_2).\end{aligned}$$

Hence $((f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) = f_0 \times f_1 \times f_2$ since these morphisms coincide after composing with each of the projections.

For the middle left triangle we have

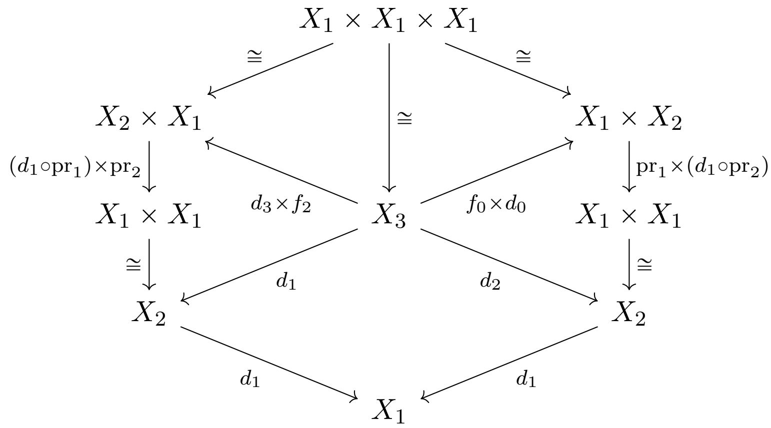
$$\begin{aligned}\mathrm{pr}_1 \circ ((d_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= (d_1 \circ \mathrm{pr}_1) \circ (d_3 \times f_2)\\ &= d_1 \circ d_3 = f_0 \circ d_1 = \mathrm{pr}_1 \circ (f_0 \times f_1) \circ d_1\end{aligned}$$

since $\delta^1 \circ \phi^0 = \delta^3 \circ \delta^1 \colon [1] \to [2] \to [3]$. We also have

$$\begin{aligned}\mathrm{pr}_2 \circ ((d_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2) &= \mathrm{pr}_2 \circ (d_3 \times f_2)\\ &= f_2 = f_1 \circ d_1 = \mathrm{pr}_2 \circ (f_0 \times f_1) \circ d_1\end{aligned}$$

since $\phi^2 = \delta^1 \circ \phi^1 \colon [1] \to [2] \to [3]$. Hence the morphisms $((d_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \circ (d_3 \times f_2)$ and $(f_0 \times f_1) \circ d_1$ coincide as morphisms into the product $X_1 \times X_1$.

Inverting the isomorphisms $f_0 \times f_1 \times f_2$, $(f_0 \circ \mathrm{pr}_1) \times (f_1 \circ \mathrm{pr}_1) \times \mathrm{pr}_2$, $\mathrm{pr}_1 \times (f_0 \circ \mathrm{pr}_2) \times (f_1 \circ \mathrm{pr}_2)$ and $f_0 \times f_1$, we obtain a commutative diagram



.

In particular, the outer compositions coincide, i. e. $m_X \circ (m_X \times \mathrm{pr}_3) = m_X \circ (\mathrm{pr}_1 \times m_X)$, which means that m_X is an associative multiplication.

In order to show that e is a right unit for m, we consider the diagram

$$\begin{array}{ccccc} & & X_2 & & \\ & \overset{s_1}{\nearrow} & \big\downarrow{\scriptstyle f_0 \times f_1} & \overset{d_1}{\searrow} & \\ X_1 & \xrightarrow[\mathrm{id}\times(e\circ\pi)]{} & X_1 \times X_1 & \xrightarrow[m]{} & X_1 \end{array} .$$

Then the triangle on the right commutes by the definition of m.

On the left side we have

$$\mathrm{pr}_1 \circ (f_0 \times f_1) \circ s_1 = f_0 \circ s_1 = \mathrm{id}_{X_1}$$

since $\mathrm{id}_{[1]} = \sigma^1 \circ \phi^0 \colon [1] \to [2] \to [1]$. For the second factor we have

$$\mathrm{pr}_2 \circ (f_0 \times f_1) \circ s_1 = f_1 \circ s_1 = s_0 \circ d_1$$

since $\delta^1 \circ \sigma^0 = \sigma^1 \circ \phi^1 \colon [1] \to [1]$, and $s_0 \circ d_1 = e \circ \pi$ since π is the unique map into the terminal object $\ast$ and $e \colon \ast \xrightarrow{\cong} X_0 \xrightarrow{s_0} X_1$ by definition. Hence the left triangle is also commutative since $(f_0 \times f_1) \circ s_1$ and $\mathrm{id}_{X_1} \times (e \circ \pi)$ agree on both factors.

The commutativity of the above diagram yields $m \circ (\mathrm{id}_{X_1} \times (e \circ \pi)) = d_1 \circ s_1 = \mathrm{id}_{X_1}$, where the latter equality is a simplicial identity. Hence e is indeed a right unit for m.

Now one can analogously show that the diagram

$$\begin{array}{ccccc} & & X_2 & & \\ & \overset{s_2}{\nearrow} & \big\downarrow{\scriptstyle f_0 \times f_1} & \overset{d_1}{\searrow} & \\ X_1 & \xrightarrow[(e\circ\pi)\times\mathrm{id}]{} & X_1 \times X_1 & \xrightarrow[m]{} & X_1 \end{array}$$

is also commutative. Hence e is also a left unit for m. □

Next, we want to see that this assignment is functorial.

Lemma A.7. *Let X, Y be special simplicial objects in $\mathscr{C}$, and $\gamma \colon X \to Y$ a morphism of simplicial objects, i. e. a natural transformation between the functors $X, Y \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$.*

Then the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\gamma_n} & Y_n \\ {\scriptstyle \prod_{i=0}^{n-1} f_i^X}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i=0}^{n-1} f_i^Y} \\ (X_1)^n & \xrightarrow[\prod_{i=0}^{n-1}(\gamma_1 \circ \mathrm{pr}_i)]{} & (Y_1)^n \end{array}$$

is commutative for all $n \in \mathbb{N}$.

Proof. For $n = 0$ the statement follows from the fact that there is a unique morphism between the terminal objects X_0 and Y_0.

For $n > 0$ we check the equality componentwise. Indeed, for $i_0 \in \{0, \dots, n-1\}$ we have

$$\begin{aligned}
\mathrm{pr}_{i_0} \circ \left(\prod_{i=0}^{n-1} f_i^Y\right) \circ \gamma_n \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} &= f_{i_0}^Y \circ \gamma_n \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} \\
&= \gamma_1 \circ f_{i_0}^X \circ \left(\prod_{i=0}^{n-1} f_i^X\right)^{-1} \\
&= \gamma_1 \circ \mathrm{pr}_{i_0} \\
&= \mathrm{pr}_{i_0} \circ \left(\prod_{i=0}^{n-1} (\gamma_1 \circ \mathrm{pr}_i)\right),
\end{aligned}$$

where the second equality follows from the naturality of γ. □

Proposition A.8. *Let $\gamma\colon X \to Y$ be a morphism between special simplicial objects.*

Then $\gamma_1\colon X_1 \to Y_1$ is a morphism of monoid objects.

Proof. For compatibility with units consider the diagram

$$\begin{array}{ccc}
* & \xrightarrow{\mathrm{id}} & * \\
{\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\
X_0 & \xrightarrow{\gamma_0} & Y_0 \\
{\scriptstyle s_0^X}\big\downarrow & & \big\downarrow{\scriptstyle s_0^Y} \\
X_1 & \xrightarrow{\gamma_1} & Y_0
\end{array} .$$

The upper square commutes since there is a unique morphism between two terminal objects. The lower square commutes by the naturality of γ. Hence γ_1 is compatible with the unit.

For compatibility with the multiplication we consider the diagram

$$\begin{array}{ccc}
X_1 \times X_1 & \xrightarrow{(\gamma_1 \circ \mathrm{pr}_1) \times (\gamma_1 \circ \mathrm{pr}_2)} & Y_1 \times Y_1 \\
\big\downarrow{\scriptstyle (f_0^X \times f_1^X)^{-1}} & & {\scriptstyle (f_0^Y \times f_1^Y)^{-1}}\big\downarrow \\
X_2 & \xrightarrow{\gamma_2} & Y_2 \\
\big\downarrow{\scriptstyle d_1^X} & & {\scriptstyle d_1^Y}\big\downarrow \\
X_1 & \xrightarrow{\gamma_1} & Y_1
\end{array} \qquad (m_X \colon X_1 \times X_1 \to X_1,\; m_Y \colon Y_1 \times Y_1 \to Y_1) .$$

The outer "triangles" commute by the definition of m_X resp. m_Y. The upper rectangle commutes by the previous lemma and the lower rectangle by the naturality of γ. Hence $\gamma_1 \circ m_X = m_Y \circ ((\gamma_1 \circ \mathrm{pr}_1) \times (\gamma_1 \circ \mathrm{pr}_2))$, i.e. γ_1 is compatible with the multiplication. □

Proposition A.6 and Proposition A.8 can be summarized as follows:

Corollary A.9. *The functor* $(\_)_1\colon (\mathbf{s}\mathscr{C})^{\mathrm{sp}} \to \mathscr{C}$ *factors through the category* $\mathscr{C}$-**Mon** *of monoid objects in* $\mathscr{C}$*.*

In fact, the following stronger statement holds.

Remark A.10. Let $\mathscr{C}$ be a category with finite products. Then $(\_)_1\colon (\mathbf{s}\mathscr{C})^{\mathrm{sp}} \to \mathscr{C}$-**Mon** is an equivalence of categories.

Indeed, a quasi-inverse is given as follows:

For $M \in \mathrm{ob}(\mathscr{C}\text{-}\mathbf{Mon})$ with "multiplication" $m\colon M \times M \to M$ and "unit" $e\colon * \to M$, we define a (special) simplicial object X^M with $X_n^M = M^n$ for $n \in \mathbb{N}$, where the structure morphisms are given by

$$d_i^{X^M} = \begin{cases} \prod_{j=2}^{n} \mathrm{pr}_j & i = 0 \\ \prod_{j=0}^{n-1} \mathrm{pr}_j & i = n \\ \left(\prod_{j=1}^{i-2} \mathrm{pr}_j\right) \times m \times \left(\prod_{j=i+1}^{n} \mathrm{pr}_j\right) & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}_{>0}$ and $0 \leqslant i \leqslant n$ resp.

$$s_i^{X^M} = \left(\prod_{j=1}^{i} \mathrm{pr}_j\right) \times (e \circ \pi) \times \left(\prod_{j=i+1}^{n} \mathrm{pr}_j\right)$$

for $n \in \mathbb{N}$.

Given a morphism $f\colon M \to N$ of monoid objects in $\mathscr{C}$, we let $\gamma^f\colon X^M \Rightarrow X^N$ be given by

$$\gamma_n^f\colon X_n^M = M^n \xrightarrow{\prod_{i=0}^{n}(f \circ \mathrm{pr}_i)} N^n = X_n^N$$

for all $n \in \mathbb{N}$.

Since we do not use this statement we omit the tedious proof of the fact that the given is assignment is a well-defined functor which is a quasi-inverse for $(\_)_1\colon (\mathbf{s}\mathscr{C})^{\mathrm{sp}} \to \mathscr{C}$-**Mon**.

B Additive Categories

In this appendix we discuss certain descriptions of additive categories which lead to the additivity of stable derivators (see Corollary 3.2).

Additive categories are covered, for example, in [9, Chapter VIII]. Here we will follow slightly different conventions which occur in [5, Section 2]. Furthermore, we will need the concepts of abelian monoid objects and abelian group objects in a category with finite products, whose definitions are similar to that of group objects (cf. [2] and [9, Section III.6]).

We begin with basic definitions and notations.

Definition B.1. A *preadditive category* is a category $\mathscr{A}$ s. t.

- $\mathscr{A}$ is pointed,
- binary (and hence all finite) products and coproducts exist in $\mathscr{A}$,
- for any $X, Y \in \operatorname{ob} \mathscr{A}$, the morphism

  $$(\mathrm{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \mathrm{id}_Y) \colon X \amalg Y \to X \times Y$$

  is an isomorphism, where $0_{X,Y} \colon X \to 0 \to Y$ resp. $0_{Y,X} \colon Y \to 0 \to X$ is the unique morphism which factors through a zero object.

Notation B.2.

- Biproducts in the above sense will be denoted by $\_\oplus\_$.
- If X, Y, X' resp. Y' are objects of a preadditive category and $f_{X,X'} \colon X \to X'$, $f_{Y,X'} \colon Y \to X'$, $f_{X,Y'} \colon X \to Y'$ resp. $f_{Y,Y'} \colon Y \to Y'$ are some morphisms, then we denote the morphism

  $$(f_{X,X'} \times f_{X,Y'}) \amalg (f_{Y,X'} \times f_{X,X'}) \colon X \oplus Y \to X' \oplus Y'$$

  by

  $$\begin{pmatrix} f_{X,X'} & f_{Y,X'} \\ f_{X,Y'} & f_{Y,Y'} \end{pmatrix}.$$

  Note that, using the universal properties of products and coproducts, any morphism $f \colon X \oplus Y \to X' \oplus Y'$ can be written as

  $$f = \begin{pmatrix} \mathrm{pr}_{X'} \circ f \circ \mathrm{in}_X & \mathrm{pr}_{X'} \circ f \circ \mathrm{in}_Y \\ \mathrm{pr}_{Y'} \circ f \circ \mathrm{in}_X & \mathrm{pr}_{Y'} \circ f \circ \mathrm{in}_Y \end{pmatrix}.$$

  Matrices of different sizes are constructed similarly.
- We will also use common abuses of notation such as denoting an identity morphism by 1 or a morphism that factors through a zero object by 0.

Next, we want to give an alternative description of preadditive categories.

Remark B.3. A preadditive category is enriched over the category **AbMon** of abelian monoids. Indeed, for any $X, Y \in \operatorname{ob}\mathscr{A}$, setting

$$f + g\colon X \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix}f & 0\\ 0 & g\end{pmatrix}} Y \oplus Y \xrightarrow{\begin{pmatrix}1\\ 1\end{pmatrix}} Y$$

for $f, g \in \operatorname{Hom}_{\mathscr{A}}(X, Y)$ yields an abelian monoid structure on $\operatorname{Hom}_{\mathscr{A}}(X, Y)$ with neutral element $0_{X,Y}$ and for any $X, Y, Z \in \operatorname{ob}\mathscr{A}$, the composition map

$$\_ \circ \_\colon \operatorname{Hom}_{\mathscr{A}}(Y, Z) \times \operatorname{Hom}_{\mathscr{A}}(X, Y) \to \operatorname{Hom}_{\mathscr{A}}(X, Z)$$

is bilinear w. r. t. this "addition operation".

Furthermore, a straightforward computation shows that composing morphisms corresponds to multiplying their matrix representations.

Proposition B.4. *Let $\mathscr{A}$ be a category that has finite products.*

Then $\mathscr{A}$ is preadditive if and only if it is enriched over the category **AbMon** *of abelian monoids, i. e. if all morphism sets of $\mathscr{A}$ have an abelian monoid structure s. t. composition is bilinear.*

Proof. A preadditive category has finite products by definition and Remark B.3 means that it is also enriched over **AbMon**.

Now let $\mathscr{A}$ be a category that has finite products and is enriched over **AbMon**. For $X, Y \in \operatorname{ob}\mathscr{A}$ let the "addition" in $\operatorname{Hom}_{\mathscr{A}}(X, Y)$ be denoted by $+_{X,Y}$ and its unit by $0_{X,Y}$.

$\mathscr{A}$ has in particular a terminal object $\ast$. The monoid structure on $\operatorname{Hom}_{\mathscr{A}}(\ast, \ast)$ is trivial since $\ast$ is a terminal object and any monoid with only one element is trivial. In particular, we have $\mathrm{id}_{\ast} = 0_{\ast,\ast}$.

For all $X \in \operatorname{ob}\mathscr{A}$, $\operatorname{Hom}_{\mathscr{A}}(\ast, X)$ has a monoid structure, hence is not empty. Now for any $f\colon \ast \to X$ we have $f = f \circ \mathrm{id}_{\ast} = f \circ 0_{\ast,\ast} = 0_{\ast,X}$ by the bilinearity of composition. Hence $\operatorname{Hom}_{\mathscr{A}}(\ast, X) = \{0_{\ast,X}\}$ for all $X \in \operatorname{ob}\mathscr{A}$, i.e. $\ast$ is also an initial object and therefore $\mathscr{A}$ is pointed. From now on 0 will denote a zero object in $\mathscr{A}$. Note that for any $X, Y \in \operatorname{ob}\mathscr{A}$, $0_{X,Y}$ is the unique morphism that factors through 0.

Let $X, Y \in \operatorname{ob}\mathscr{A}$. We want to endow $X \times Y$ with the structure of a coproduct of X and Y s. t. $(\mathrm{id}_X \times 0_{X,Y}) \amalg (0_{Y,X} \times \mathrm{id}_Y) = \mathrm{id}_{X\times Y}$. This enforces the structure morphisms of the coproduct to be $\mathrm{in}_1 := \mathrm{id}_X \times 0_{X,Y}$ and $\mathrm{in}_2 := 0_{Y,X} \times \mathrm{id}_Y$.

Given $Z \in \operatorname{ob}\mathscr{A}$ and morphisms $f_1\colon X \to Z$ and $f_2\colon Y \to Z$, define $f_1 \amalg f_2$ to be $(f_1 \circ \mathrm{pr}_1) +_{X\times Y,Z} (f_2 \circ \mathrm{pr}_2)$. Then we have indeed

$$\begin{aligned}
((f_1 \circ \mathrm{pr}_1) +_{X\times Y,Z} (f_2 \circ \mathrm{pr}_2)) \circ \mathrm{in}_1 &=\\
((f_1 \circ \mathrm{pr}_1) +_{X\times Y,Z} (f_2 \circ \mathrm{pr}_2)) \circ (\mathrm{id}_X \times 0_{X,Y}) &=\\
((f_1 \circ \mathrm{pr}_1) \circ (\mathrm{id}_X \times 0_{X,Y})) +_{X,Z} ((f_2 \circ \mathrm{pr}_2)) \circ (\mathrm{id}_X \times 0_{X,Y})) &=\\
(f_1 \circ \mathrm{id}_X) +_{X,Z} (f_2 \circ 0_{X,Y}) &=\\
f_1 +_{X,Z} 0_{X,Z} &= f_1
\end{aligned}$$

and similarly

$$\begin{aligned}((f_1 \circ \mathrm{pr}_1) +_{X\times Y,Z} (f_2 \circ \mathrm{pr}_2)) \circ \mathrm{in}_2 = \\ ((f_1 \circ \mathrm{pr}_1) \circ (0_{Y,X} \times \mathrm{id}_Y)) +_{Y,Z} ((f_2 \circ \mathrm{pr}_2)) \circ (0_{Y,X} \times \mathrm{id}_Y)) = \\ 0_{Y,Z} +_{Y,Z} f_2 = f_2.\end{aligned}$$

Next, we claim that

$$\mathrm{id}_{X\times Y} = (\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2).$$

Indeed, by the bilinearity of composition we have

$$\begin{aligned}\mathrm{pr}_1 \circ ((\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2)) = \\ (\mathrm{pr}_1 \circ (\mathrm{pr}_1 \times 0_{X\times Y,Y})) +_{X\times Y,X} (\mathrm{pr}_1 \circ (0_{X\times Y,X} \times \mathrm{pr}_2)) = \\ \mathrm{pr}_1 +_{X\times Y,X} 0_{X\times Y,X} = \mathrm{pr}_1\end{aligned}$$

and similarly

$$\mathrm{pr}_2 \circ ((\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2)) = \mathrm{pr}_2.$$

Hence the two morphisms coincide since they agree on both factors.

Moreover, note that we have

$$\mathrm{pr}_1 \circ (\mathrm{id}_X \times 0_{X,Y}) \circ \mathrm{pr}_1 = \mathrm{pr}_1$$

and

$$\mathrm{pr}_2 \circ (\mathrm{id}_X \times 0_{X,Y}) \circ \mathrm{pr}_1 = 0_{X,Y} \circ \mathrm{pr}_1 = 0_{X\times Y,Y},$$

which means that $\mathrm{in}_1 \circ \mathrm{pr}_1 = (\mathrm{id}_X \times 0_{X,Y}) \circ \mathrm{pr}_1 = \mathrm{pr}_1 \times 0_{X\times Y,Y}$ since these morphisms agree on both factors. Similarly, we also have $\mathrm{in}_2 \circ \mathrm{pr}_2 = (0_{Y,X} \times \mathrm{id}_Y) \circ \mathrm{pr}_2 = 0_{X\times Y,X} \times \mathrm{pr}_2$.

Now let $f'\colon X \times Y \to Z$ be another morphism s. t. $f' \circ \mathrm{in}_1 = f_1$ and $f' \circ \mathrm{in}_2 = f_2$. Then, using the bilinearity of composition, the above calculations yield

$$\begin{aligned}f' \circ \mathrm{id}_{X\times Y} &= f' \circ ((\mathrm{pr}_1 \times 0_{X\times Y,Y}) +_{X\times Y,X\times Y} (0_{X\times Y,X} \times \mathrm{pr}_2)) \\ &= (f' \circ (\mathrm{pr}_1 \times 0_{X\times Y,Y})) +_{X\times Y,X\times Y} (f' \circ (0_{X\times Y,X} \times \mathrm{pr}_2)) \\ &= (f' \circ \mathrm{in}_1 \circ \mathrm{pr}_1) +_{X\times Y,X\times Y} (f' \circ \mathrm{in}_2 \circ \mathrm{pr}_2) \\ &= (f_1 \circ \mathrm{pr}_1) +_{X\times Y,X\times Y} (f_2 \circ \mathrm{pr}_2) = f_1 \amalg f_2.\end{aligned}$$

Hence in_1 and in_2 do endow $X \times Y$ with a suitable coproduct structure. □

Using this characterization we obtain a generic class of preadditive categories.

Proposition B.5. *Let $\mathscr{C}$ be a category which has finite products.*

*Then the category $\mathscr{C}$-**AbMon** of abelian monoid objects (with homomorphisms of monoid objects between them) is a preadditive category.*

Proof. First we note that $\mathscr{C}$-**AbMon** can alternatively be described as follows:

$$\mathrm{ob}(\mathscr{C}\text{-}\mathbf{AbMon}) = \{M \in \mathrm{ob}\,\mathscr{C} \mid \mathrm{Hom}_{\mathscr{C}}(\_, M) \text{ factors through } \mathbf{AbMon}\}\,,$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}\text{-}\mathbf{AbMon}}(M,N) = \{f \in \mathrm{Hom}_{\mathscr{C}}(M,N) \mid\ & \\ & (f_*)_X\colon \mathrm{Hom}_{\mathscr{C}}(X,M) \to \mathrm{Hom}_{\mathscr{C}}(X,N) \\ & \text{is a morphisms of (abelian) monoids for all} \\ & X \in \mathrm{ob}\,\mathscr{C}\}\end{aligned}$$

for any $M, N \in \mathrm{ob}(\mathscr{C}\text{-}\mathbf{AbMon})$.

Hence, for any $M, N \in \mathrm{ob}(\mathscr{C}\text{-}\mathbf{AbMon})$, $\mathrm{Hom}_{\mathscr{C}\text{-}\mathbf{AbMon}}(M,N)$ has an abelian monoid structure given by pointwise addition and units on the level of represented functors. Then composition is bilinear w. r. t. this addition since morphisms between monoid objects are chosen to preserve the addition on the homomorphism sets.

This means that $\mathscr{C}$-**AbMon** is enriched over **AbMon** and hence a preadditive category by the previous proposition. □

Remark B.6. Remark B.3 implies also that any object X of a preadditive category has the structure of an abelian monoid object given by the codiagonal morphism $\nabla := (\,1\ 1\,) : X \oplus X \to X$ and the "unit" $0 \to X$. Dually, X has also the structure of a cocommutative comonoid object given by the diagonal morphism $\Delta := \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) : X \oplus X \to X$ and the "counit" $X \to 0$.

In fact, for a preadditive category $\mathscr{A}$, the functor $\mathscr{A} \to \mathscr{A}$-**AbMon** which endows an object with the above abelian monoid structure is an equivalence of categories. However, we will neither use nor prove this statement.

As the prefix "pre" suggests, preadditive categories are not quite what we are looking for. We get to the concept of additive categories by requiring that additive inverses of morphisms exist.

Proposition B.7. *Let $\mathscr{A}$ be a preadditive category.*
Then the following are equivalent:

(i) For any $X \in \mathrm{ob}\,\mathscr{A}$, the "shear morphism"

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} : X \oplus X \to X \oplus X$$

is an isomorphism.

(ii) For any $X \in \mathrm{ob}\,\mathscr{A}$, the identity morphism id_X has an additive inverse in $\mathrm{End}_{\mathscr{A}}(A)$.

(iii) *For any $X, Y \in \operatorname{ob}\mathscr{A}$, each $f \in \operatorname{Hom}_{\mathscr{A}}(X, Y)$ has an additive inverse.*

(iv) *For any $X \in \operatorname{ob}\mathscr{A}$, the abelian monoid object $(X, \nabla, 0 \to X)$ is an (abelian) group object.*

(v) *For any $X \in \operatorname{ob}\mathscr{A}$, the cocommutative comonoid object $(X, \Delta, X \to 0)$ is a (cocommutative) cogroup object.*

Proof. "*(i)* ⇒ *(ii)*": Let the inverse of the shear morphism of X be given by

$$\begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} : X \oplus X \to X \oplus X.$$

Then we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j_{1,1} & j_{1,2} \\ j_{2,1} & j_{2,2} \end{pmatrix} = \begin{pmatrix} j_{1,1} + j_{2,1} & j_{1,2} + j_{2,2} \\ j_{2,1} & j_{2,2}. \end{pmatrix}$$

Hence $j_{1,2} = 0_{X,X}$ and $j_{1,1} = j_{2,2} = \mathrm{id}_X$. This yields

$$\mathrm{id}_X + j_{1,2} = j_{2,2} + j_{1,2} = 0_{X,X},$$

so $j_{1,2}$ is an additive inverse of id_X.

"*(ii)* ⇒ *(iii)*": Let $-\mathrm{id}_X$ be an additive inverse for id_X. Then the bilinearity of composition yields

$$f + f \circ (-\mathrm{id}_X) = f \circ \mathrm{id}_X + f \circ (-\mathrm{id}_X) = f \circ (\mathrm{id}_X + (-\mathrm{id}_X)) = f \circ 0_{X,X} = 0_{X,Y},$$

i. e. $f \circ (-\mathrm{id}_X)$ is an additive inverse for f.

"*(iii)* ⇒ *(iv)*": Note that X is a group object in $\mathscr{A}$ if and only if its represented functor $\operatorname{Hom}_{\mathscr{A}}(\_, X)$ factors through the category **Grp** of groups. Since X is an abelian monoid object, we already know that $\operatorname{Hom}_{\mathscr{A}}(\_, X)$ factors through **AbMon**. Now the fact that for each $Y \in \operatorname{ob}\mathscr{A}$ each $f \in \operatorname{Hom}_{\mathscr{A}}(Y, X)$ has an additive inverse implies that the abelian monoids $(\operatorname{Hom}_{\mathscr{A}}(Y, X), +_{Y,X}, 0_{Y,X})$ are in fact abelian groups. Since all monoid homomorphisms between groups are already homomorphisms of groups, this means that $\operatorname{Hom}_{\mathscr{A}}(\_, X)$ factors through the category of (abelian) groups.

"*(iv)* ⇒ *(v)*": If X is a group object with the "multiplication" given by ∇, there exists a morphism $j \colon X \to X$ s. t.

$$0_{X,X} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} = \mathrm{id}_X \circ \mathrm{id}_X + \mathrm{id}_X \circ j = \mathrm{id}_X + j.$$

Hence for comonoid structure on X we obtain

$$\begin{pmatrix} 1 & j \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathrm{id}_X \circ \mathrm{id}_X + j \circ \mathrm{id}_X = \mathrm{id}_X + j = 0_{X,X}.$$

A similar argument shows that the fact that j is also a "left inverse for the multiplication of X" implies that j is also a "left inverse for the comultiplication of X". In total, we obtain that $(X, \Delta, X \to 0, j)$ is a cogroup object.

"$(v) \Rightarrow (i)$": Let $j\colon X \to X$ be the "coinverse" morphism w. r. t. Δ. Then calculations similar to the ones in the proof of the previous implication yield that $\mathrm{id}_X + j = 0_{X,X} = j + \mathrm{id}_X$. Hence we obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & j+1 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+j \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathrm{id}_{X \oplus X} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we see that the shear morphism is an isomorphism. □

Definition B.8. A preadditive category is called *additive* if it satisfies one (and hence all) of the conditions in the previous proposition.

Additive categories also have an alternative characterization similar to the one in Proposition B.4 for preadditive categories.

Corollary B.9. *A category $\mathscr{A}$ which has finite products is additive if and only if it is enriched over the category* **Ab** *of abelian groups.*

Proof. If $\mathscr{A}$ is additive then the condition (iii) in Proposition B.7 means that $\mathscr{A}$ is enriched not only over **AbMon**, but even over **Ab** since additive inverses exist in homomorphism monoids and (bi)linear maps of abelian groups are exactly (bi)linear maps of underlying monoids.

Conversely, if $\mathscr{A}$ is enriched over **Ab**, then $\mathscr{A}$ is preadditive by Proposition B.4 and the condition (iii) in Proposition B.7 is fulfilled since additive inverses in all homomorphism monoids exist. □

This characterization yields a generic class of examples which is used in the proof of Corollary 3.2.

Corollary B.10. *Let $\mathscr{C}$ be a category which has finite products.*

*Then the category $\mathscr{C}$-**Ab** of abelian group objects (with homomorphisms of group objects between them) is an additive category.*

Proof. Like the previous corollary, this follows immediately from Proposition B.4 and the condition (iii) in Proposition B.7. □

Lastly, let us mention that, similar to the case of preadditive categories, an additive category $\mathscr{A}$ is in fact equivalent to the category of abelian group objects in $\mathscr{A}$.

References

[1] Cisinski, D.-C. Images directes cohomologiques dans les catégories de modèles. *Ann. Math. Blaise Pascal 10*, 2 (2003), 195–244.

[2] Eckmann, B., and Hilton, P. J. Group-like structures in general categories. I. Multiplications and comultiplications. *Math. Ann. 145* (1962), 227–255.

[3] Franke, J. Uniqueness theorems for certain triangulated categories with an Adams spectral sequence. Preprint, August 8, 1996, K-theory Preprint Archives, `http://www.math.uiuc.edu/K-theory/0139/`.

[4] Groth, M. Derivators, pointed derivators and stable derivators. *Algebr. Geom. Topol. 13*, 1 (2013), 313–374.

[5] Groth, M. Selected Topics in Topology: Derivators. `http://guests.mpim-bonn.mpg.de/mgroth/teaching/derivators14/Groth_derivators-NOTES.pdf`, 2015. Accessed version published on 2015-01-27.

[6] Grothendieck, A. Les Dérivateurs. `http://webusers.imj-prg.fr/~georges.maltsiniotis/groth/Derivateurs.html`, 1991. Accessed version published on 2014-12-24.

[7] Heller, A. Homotopy theories. *Mem. Am. Math. Soc. 383* (1988), 78.

[8] Hovey, M. *Model categories.* Providence, RI: American Mathematical Society, 1999.

[9] Mac Lane, S. *Categories for the working mathematician. 2nd ed.*, 2nd ed ed. New York, NY: Springer, 1998.

[10] Segal, G. Classifying spaces and spectral sequences. *Publ. Math., Inst. Hautes Étud. Sci. 34* (1968), 105–112.