Spectral Theorem for Unbounded Operators

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Einleitung

Ein Spektraltheorem oder Spektralkalkl gibt der Anschauung, dass man Operatoren auf einem Hilbertraum in Funktionen einsetzen kann, eine rigorose mathematische Grundlage. Dass man Operatoren in Polynome einsetzen kann, und immer noch sinnvolle Ausdrcke entstehen, liegt auf der Hand. Wie verhlt es sich jedoch mit stetigen oder gar messbaren Funktionen? Ergibt der Ausdruck $f(T)$ fr beliebige Funktionen f und Operatoren T berhaupt Sinn? Was ist der Definitionsbereich und Bild von T? Ist $f(T)$ dicht definiert? Diese Fragen mchte ich in dieser Arbeit, soweit es geht, beantworten.

Das Ziel dieser Arbeit ist ein Spektraltheorem fr unbeschrnkte normale Operatoren. Aus diversen Vorlesungen an der Universitt Bonn waren mir Spektraltheoreme fr explizite Klassen von Operatoren bekannt, zum Beispiel fr kompakte, selbstadjungerte Operatoren. Als ich in einem Seminar eine Variante fr unbeschrnkte Operatoren benutzen musste, entschied ich mich mehr mit diesem Thema zu beschftigen. Diese Arbeit ist an Studenten der Mathematik oder Physik gerichtet, welche eine mathematisch rigorose Formulierung des Spektraltheorems fr unbeschrnkte Operatoren kennen lernen mchten.

In meiner Bachelorarbeit wird in Kapitel 2 mit Hilfe des Gelfandschen Transformationsatzes ein Spektraltheorem fr beschrnkte normale Operatoren bewiesen. In Kapitel 3 wird versucht, die Methoden des voran gegangenen Kapitels auf unbeschrnkte Operatoren zu erweitern. Um dies zu tun, muss das Spektraltheorem auf messbare Funktionen erweitert werden. Dazu wird in Kapitel 4 beschrieben wie sich Erweiterungen des Spektraltheorems auf messbare Funktionen verhalten, das heit, welche Klassen von Operatoren erhalten werden. Abschlieend werden die Ergebnisse der vorherigen Kapitel benutzt, um in Kaptiel 5 das Spektraltheorem fr unbeschrnkte normale Operatoren zu beweisen, und in Kapitel 6 einige Anwendungen aufgezeigt.

In der Literatur werden zum Beweis von Spektraltheoremen oft operatorwertige Mae auf bestimmten σ–Algebren genutzt. Dies wird in dieser Arbeit explizit nicht genutzt. Welche der Mglichkeiten man benutzt, bleibt der eigenen Vorliebe berlassen. Der Riesz–Markov–Kakutani Darstellungssatz ([5, Theorem 6.3.4]) zeigt, dass beide Herangehensweisen lediglich zwei Seiten der gleichen Medaille sind.

Ich mchte mich an dieser Stelle bei meinem Betreuer Herrn Professor Lesch bedanken. Erst durch seine Betreuung und stetige Erreichbarkeit wurde diese Arbeit ermglicht. Weiterhin danke ich meinen Kommilitonen fr das Korrekturlesen dieser Arbeit. Zum Schluss mchte ich meinen Eltern fr die Untersttzung meines Studiums danken. Ohne Sie wre diese Arbeit nie zu Stande gekommen.

Introduction

A spectral theorem or spectral calculus is a mathematical justification of the idea that one can plug operators into functions. One can easily see that for polynomials, it makes sense to plug in operators. But what about continuous or even measurable functions? Does the expression $f(T)$ make sense for arbitrary operators? What is the domain and range of the corresponding operators? Is f densely defined? In this text, I want to answer these questions.

The goal of this thesis is to develop a spectral theorem for unbounded normal operators. From different lectures at the university of Bonn, I knew a few explicit formulations of spectral theorems about compact self-adjoint operators. Preparing for a seminar, I was forced to work with a spectral theorem for unbounded operators, and decided that I want to understand this subject properly. This thesis is aimed at students of mathematics and physics, who want to see a rigorous statement of the spectral theorem for unbounded operators.

In Chapter 2, we will construct a continuous spectral theorem for bounded operators using the Gelfand representation theorem. Following this, we try to extend this to unbounded operators in Chapter 3. To do this, we have to extend the spectral calculus to measurable functions, done in Chapter 4. In Chapter 5, we use the previous results to develop a spectral calculus for unbounded normal operators. The last chapter will show a few applications.

To prove a spectral theorem, one often encounters projection valued measures. We will not use these in this thesis, but rather use operator valued functionals. However, it is irrelevant what specific tool you choose: By the Riesz–Markov–Kakutani representation theorem, both are equivalent choices.

At this point, I want to thank Professor Lesch. His constant availability made this thesis possible. Furthermore, I want to thank my fellow students for proof-reading this work. In the end, I thank my parents for always supporting me. Without them, this work would not even have been started.

1 Preliminaries

In this section, I want to state a few theorems and definitions, which will be used later on. If no proof is given, a reference will be stated nevertheless.

For an unital algebra $\mathfrak{A}$, we denote the invertible elements by $\mathbf{GL}(\mathfrak{A})$.

Definition 1.1 (Spectrum)**.** Let $\mathfrak{A}$ be a unital commutative Banach algebra. For A an element of $\mathfrak{A}$, the *spectrum* of A in $\mathfrak{A}$, is defined as

$$\mathrm{Sp}_{\mathfrak{A}}(A) := \{z \in \mathbb{C} \mid A - zI \notin \mathbf{GL}(\mathfrak{A})\}\,.$$

The *spectrum* of $\mathfrak{A}$ is defined as

$$\mathrm{Sp}(\mathfrak{A}) := \{\chi \colon \mathfrak{A} \to \mathbb{C} \mid \chi \in \mathrm{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}(\mathfrak{A}, \mathbb{C}), \chi \neq 0\}\,.$$

Proposition 1.2. *Using the notation above,*

$$\mathrm{Sp}_{\mathfrak{A}}(A) = \big\{\chi(A) \bigm| \chi \in \mathrm{Sp}\,\mathfrak{A}\big\}\,.$$

In other words, the spectrum of an element is the image of that particular element under the spectrum of the algebra.

The proof can be found in [5, Ch. 4.2]. By $\mathfrak{A}'$ we denote the dual space $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{A}, \mathbb{C})$, endowed with the weak-$*$ topology.

Theorem 1.3 (Gelfand–Naimark)**.** *Let $\mathfrak{A}$ be a commutative C^*-algebra. If $\mathrm{Sp}(\mathfrak{A})$ is equipped with the subspace topology of $\mathfrak{A}'$, it becomes a compact space, with a canonical isometric, involutive, surjective algebra homomorphism*

$$\mathcal{G} \colon \mathfrak{A} \to \mathrm{C}(\mathrm{Sp}(\mathfrak{A})),\ A \mapsto (\hat{A} := \mathcal{G}(A) \colon\ \mathrm{Sp}(\mathfrak{A}) \to \mathbb{C},\ \gamma \mapsto \gamma(A)).$$

The proof can be found in [5, Ch. 4.3]. The map in the theorem is the so called Gelfand transform.

All Hilbert spaces are assumed to be separable. The scalar products are conjugate linear in the first variable. All spaces shall be second countable and Hausdorff. The separability and countability assumptions are not necessary in most proofs, but simplify notation.

The reader not familiar with closed operators is advised to revisit the basic definitions found in [1, Ch. 10]. When we speak of an operator, we always mean a linear operator. Let T be such an operator on some Hilbert space $\mathcal{H}$. We will adopt the notation from [1, Ch. 10], that is, T is not even assumed to be defined for any non-zero element. Later on, we will only concern ourselves with densely defined, closed operators. By $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ we denote the domain and the range of the operator. The set of bounded linear operators will be called $\mathcal{B}(\mathcal{H})$, in contrast to $\mathscr{L}(\mathcal{H})$, which is the set of all linear operators.

Definition 1.4. If T and S are operators on $\mathcal{H}$, we say T *extends* S, if $\mathfrak{D}(S) \subset \mathfrak{D}(T)$ and $Sx = Tx$ for all x in $\mathfrak{D}(S)$. We write $S \subset T$.

Let S and T be operators on $\mathcal{H}$. By $S + T$, we denote the operator with domain $\mathfrak{D}(S) \cap \mathfrak{D}(T)$ and rule $(S + T)x = Sx + Tx$. Note that this does not give $\mathscr{L}(\mathcal{H})$ the structure of a vector space, for $(S + T) + (-T) \neq S$, because $\mathfrak{D}(T) \cap \mathfrak{D}(S) \neq \mathfrak{D}(S)$. TS is defined to be the operator with domain $S^{-1}\mathfrak{D}(T)$. The reader should be aware, that with the just defined operations, $\mathscr{L}(\mathcal{H})$ does not admit the structure of an algebra, and as previously remarked, not even that of a vector space. This is one of the reasons, why one has to be careful when working with unbounded operators.

Since closed operators are not defined on all of $\mathcal{H}$, there is no obvious notion of an inverse to such an operator. However, we have the following

Definition 1.5. Let T denote a closed operator on $\mathcal{H}$. We say that T is *boundedly invertible* if $T\colon \mathfrak{D}(T) \to \mathfrak{R}(T) = \mathcal{H}$ is a bijection, and $T^{-1}\colon \mathcal{H} \to \mathfrak{D}(T) \subset \mathcal{H}$ is continuous. T^{-1} is called the *bounded inverse* of T. As T^{-1} is continuous, the closed graph theorem implies the boundedness of T^{-1}.

Remark 1.6. If T is boundedly invertible, then the inverse is unique.

For $A, B, \ldots \in \mathfrak{A}$, we denote by $\langle A, B, \ldots \rangle$ the C^*-subalgebra of $\mathfrak{A}$, generated by the elements $A, B, \ldots$

2 Spectral Theorem for Bounded Opearators

In this chapter, we use the Gelfand transform to prove a continuous spectral theorem for bounded normal operators, and some auxiliary results about the spectrum of C^*-algebras.

Proposition 2.1. *Let $\mathfrak{A}$ be a unital C^*-subalgebra of the C^*-algebra $\mathfrak{B}$. Let A be an element of $\mathfrak{A}$, which is invertible in $\mathfrak{B}$. Then A is already invertible in $\mathfrak{A}$. In other words*

$$\mathrm{Sp}_{\mathfrak{A}}(A) = \mathrm{Sp}_{\mathfrak{B}}(A) \text{ for all } A \text{ in } \mathfrak{A}.$$

Proof. First assume $A = A^*$. We have $\mathrm{Sp}_{\mathfrak{A}}(A) \subset \mathbb{R}$, implying that $(A + i\lambda I)$ is invertible in $\mathfrak{A}$, for all $\lambda \neq 0$. As

$$\lim_{\lambda \to 0} (A + i\lambda I) = A,$$

by continuity of the inverse map, and the assumption that A is invertible, we get

$$\lim_{\lambda \to 0} (A + i\lambda I)^{-1} = A^{-1}.$$

Because $(A + i\lambda I)^{-1}$ is an element of $\mathfrak{A}$ for all $\lambda \neq 0$, the statement holds for self-adjoint A, as $\mathfrak{A}$ is a closed subalgebra.

For more general A, we consider the self-adjoint element A^*A, with inverse $(A^*A)^{-1} = A^{-1}(A^{-1})^*$. Since $\mathfrak{A}$ is an involutive algebra, A^*A is an element of $\mathfrak{A}$, which implies that A is left-invertible in $\mathfrak{A}$, with inverse $(A^*A)^{-1}A^*$. Using the same argument with the normal element AA^*, one gets the right-invertibility of A. Thus A is invertible, and both inverses coincide. □

Corollary 2.2. *Let $\mathfrak{A}$ be a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, $T \in \mathfrak{A}$. Then*

$$\mathrm{Sp}_{\mathfrak{A}}(T) = \mathrm{Sp}_{\mathcal{B}(\mathcal{H})}(T) = \mathrm{Sp}(T).$$

Proposition 2.3 (Functional calculus for normal elements)**.** *Let $\mathfrak{B}$ be a C^*-algebra with unit, and A a normal element. Then the algebra $\mathfrak{A} = \langle A, I \rangle$ generated by A and the identity I, is a commutative involutive subalgebra, which is isomorphic to $\mathrm{C}(\mathrm{Sp}\, A)$, where $\mathrm{Sp}(A)$ is identified with $\mathrm{Sp}(\mathfrak{A})$ via the Gelfand-transform.*

Proof. First, we show that $\mathcal{G}_A \colon \mathrm{Sp}(\mathfrak{A}) \to \mathrm{Sp}(A) \subset \mathbb{C}$ is injective. Let χ_1, χ_2 be elements of $\mathrm{Sp}(\mathfrak{A})$. If $\mathcal{G}_A(\chi_1) = \mathcal{G}_A(\chi_2) = \chi_2(A) = \chi_1(A)$, then $\chi_1(A^*) = \chi_2(A^*)$ also. Since $\chi_1(I) = \chi_2(I) = 1$, we see that $\chi_1 = \chi_2$ on all polynomials in A and A^*. Because χ_1, χ_2 are continous, they have to coincide on $\mathfrak{A}$.

By Proposition 1.2, $\mathcal{G}_A$ is surjective. Hence, $\mathcal{G}_A$ is a continous bijection from $\mathrm{Sp}(\mathfrak{A})$ to $\mathrm{Sp}(A)$. As $\mathrm{Sp}(\mathfrak{A})$ is compact and $\mathrm{Sp}(A)$ Hausdorff, $\mathcal{G}_A$ is a homeomorphism. By the theorem of Gelfand–Neimark

$$\mathcal{G} \colon \mathfrak{A} \to \mathrm{C}(\mathrm{Sp}\, \mathfrak{A})$$

is an isomorphism. We get the following commutative diagram, which yields the result:

$$
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{B \mapsto \mathcal{G}_B} & \mathrm{C}(\operatorname{Sp}\mathfrak{A}) \\
& \searrow^{B \mapsto \mathcal{G}_B \circ \mathcal{G}_A^{-1}} & \downarrow{\scriptstyle \mathcal{G}_B \mapsto \mathcal{G}_B \circ \mathcal{G}_A^{-1}} \\
& & \mathrm{C}(\operatorname{Sp} A)
\end{array}
.
$$

□

Remark 2.4. Let $\Phi\colon \mathrm{C}(\operatorname{Sp} A) \to \mathfrak{A}$ be the inverse of the isomorphism from the previous theorem defined by $B \mapsto \mathcal{G}_B \circ \mathcal{G}_A^{-1}$. For $f \in \mathrm{C}(\operatorname{Sp} A)$, we get

$$\Phi(f) = \mathcal{G}^{-1}(f \circ \mathcal{G}_A).$$

Thus, one retrieves the generators of $\mathfrak{A}$ via

$$
\begin{aligned}
\Phi(1_{\mathrm{Sp}(A)}) &= \mathcal{G}^{-1}(1_{\mathrm{Sp}(A)} \circ \mathcal{G}_A) \\
&= \mathcal{G}^{-1}(1_{\mathrm{Sp}(\mathfrak{A})}), \\
\Phi(\mathrm{id}_{\mathrm{Sp}(A)}) &= \mathcal{G}^{-1}(\mathrm{id}_{\mathrm{Sp}(A)} \circ \mathcal{G}_A) \\
&= \mathcal{G}^{-1}(\mathcal{G}_A) = A, \\
\Phi(\overline{\mathrm{id}}_{\mathrm{Sp}(A)}) &= A^*.
\end{aligned}
$$

The map Φ gives us the possibility to identify functions on the closure of polynomials in $z, \overline{z}$ on $\mathrm{Sp}(A)$ with elements in A. By the theorem of Stone-Weierstrass, the closure of polynomials in $z, \overline{z}$ on $\mathrm{Sp}(A)$ are all continuous functions on $\mathrm{Sp}(A)$. Furthermore Φ is completely determined by its values on $1_{\operatorname{Sp} A}$ and $id_{\operatorname{Sp} A}$.

Example 2.5. Let $\mathfrak{B} = \mathcal{B}(\mathcal{H})$ be the space of bounded linear operators on some Hilbert space $\mathcal{H}$, T a normal element and $\mathfrak{A}$ the C^*-algebra generated by T. Any entire function $f\colon \mathbb{C} \to \mathbb{C}$ is continuous on $\mathrm{Sp}(A)$, and hence gives us an element $f(A)$ in $\mathfrak{A}$.

In general, a complex square root does not give a holomorphic function on $\mathrm{Sp}(A)$. However for self-adjoint A, we can still define a continuous square root, as $\mathrm{Sp}(A)$ consists only of real numbers.

$$
\sqrt{\ }\colon \mathrm{Sp}(A) \to \mathbb{C},\ z \mapsto \begin{cases} \sqrt{z} & \text{if } z > 0, \\ i\sqrt{-z} & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}
$$

If the normal operator T is invertible, 0 is not an element of the spectrum. Since the spectrum is a closed subset of $\mathbb{C}$, there is a neighborhood U containing 0, such that U does not intersect $\mathrm{Sp}(T)$. Thus $f(x) = {}^1\!/_x$ is a continuous function on $\mathrm{Sp}(T)$. Since we have $1 = xf(x)$, the spectral theorem implies that f corresponds to T^{-1}.

Proposition 2.6. *Let $\mathfrak{B}$ be an involutive, unital Banach algebra, $\mathfrak{A}$ a unital C^*-algebra, and let*

$$\Phi\colon \mathfrak{B} \to \mathfrak{A}$$

be an involutive algebra homomorphism. Then Φ is continuous and norm decreasing.

Proof. Let $B \in \mathfrak{B}$. We have

$$\mathrm{Sp}_{\mathfrak{A}}(\Phi(B)) \subset \mathrm{Sp}_{\mathfrak{B}}(B).$$

For the spectral radius, one therefore has

$$\rho(\Phi(B)) \leq \rho(B) \leq \|B\|.$$

Consequently

$$\begin{aligned} \|\Phi(B)\|^2 &= \|(\Phi(B))^* \Phi(B)\| \\ &= \|\Phi(B^* B)\| \\ &= \rho(\Phi(B^* B)) \leq \|B^* B\| \leq \|B\|^2. \end{aligned}$$

This gives $\|\Phi\| \leq 1$. □

Corollary 2.7. *Using the same notation as before, the isomorphism*

$$\Phi \colon \mathrm{C}(\mathrm{Sp}\,\mathfrak{A}) \to \mathfrak{A}\ ,\ f \mapsto \mathcal{G}^{-1}(f \circ \mathcal{G}_A)$$

is the only C^*-algebra homomorphism with the property that

$$\Phi(1_{\mathrm{Sp}\,A}) = I \text{ and } \Phi(\mathrm{id}_{\mathrm{Sp}\,A}) = A.$$

Proof. If $\Psi \colon \mathrm{C}(\mathrm{Sp}\,A) \to \mathfrak{A}$ is another algebra homomorphism with the properties above, then $\Psi = \Phi$ on all polynomials in z and $\overline{z}$ on $\mathrm{Sp}(A)$. By Proposition 2.6, we know that both homomorphisms are continuous. Thus they must coincide on $\mathrm{C}(\mathrm{Sp}\,A)$, by the theorem of Stone-Weierstrass. □

3 Unbounded Operators

Since we have a functional calculus for normal bounded operators, one might hope that we can extend our results to unbounded operators. But the previous result relied on the Gelfand transform, which in turn relied on the existence of certain structures, such as the operator being an element in an algebra. But as previously remarked, closed operators are not that nice. This chapter follows [4]. Lemma 3.3 is taken from [1, p. 319].

From now on, let $T \in \mathscr{L}(\mathcal{H})$ be a closed normal operator. We endow $\mathfrak{D}(T)$ with the graph scalar product, where $\iota\colon \mathfrak{D}(T) \hookrightarrow \mathcal{H}$ is the canonical inclusion.

$$\langle x, y \rangle_T \coloneqq \langle \iota x, \iota y \rangle_{\mathcal{H}} + \langle T\iota x, T\iota y \rangle_{\mathcal{H}} \,,$$

This turns $\mathfrak{D}(T)$ into a Hilbert space. The topology given by the graph scalar product is finer than the subspace topology, as convergence in the graph norm implies convergence in the subspace topology. Furthermore, T seen as a map from $(\mathfrak{D}(T), \langle \cdot, \cdot \rangle_T)$ to $\mathcal{H}$, is continous.

If there is no room for misinterpretation, we will omit the $\mathcal{H}$ in the scalar product. The adjoint of T as a closed operator from $\mathcal{H}$ to itself, will be called T^*.

Lemma 3.1. *$\mathfrak{D}(T^*T)$ is dense in $(\mathfrak{D}(T), \langle \cdot, \cdot \rangle_T)$.*

Proof. We show that $\mathfrak{D}(T^*T)^{\perp_T} = 0$. Let $x \in \mathfrak{D}(T^*T),\ y \in \mathfrak{D}(T^*T)^{\perp_T}$. Then

$$\begin{aligned} 0 = \langle x, y \rangle_T &= \langle \iota x, \iota y \rangle_{\mathcal{H}} + \langle T\iota x, T\iota y \rangle_{\mathcal{H}} \\ &= \langle \iota x, \iota y \rangle_{\mathcal{H}} + \langle T^*T\iota x, \iota y \rangle_{\mathcal{H}} = \langle (I + T^*T)\iota x, \iota y \rangle_{\mathcal{H}} \,. \end{aligned}$$

Therefore, if we prove $\mathfrak{R}(I + T^*T)$ is dense in $\mathcal{H}$, the claim is proven as well. Since $\mathfrak{R}(I + T^*T)^{\perp_{\mathcal{H}}} = \ker(I + T^*T)$, we prove injectivity of $(I + T^*T)$,

$$\begin{aligned} \|(I + T^*T)x\|^2 &= \|x\|^2 + \|T^*Tx\|^2 + \langle x, T^*Tx \rangle + \langle T^*Tx, x \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + 2\,\langle Tx, Tx \rangle \\ &= \|x\|^2 + \|T^*Tx\|^2 + 2\|Tx\|^2 \geq \|x\|^2. \end{aligned}$$

□

We have two ways to interpret the inclusion map ι:

1. as an operator on $\mathcal{H}$, namely the identity with domain $\mathfrak{D}(T)$, or
2. as a bounded linear operator $\iota\colon (\mathfrak{D}(T), \langle \cdot, \cdot \rangle_T) \to (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$.

Using the second interpretation, we get

Proposition 3.2. *$(I + T^*T)$ is boundedly invertible, with inverse $\iota\iota^*$.*

Proof. The calculation of the last lemma shows that $(I + T^*T)$ is injective. Therefore, it suffices to show that there exists a right inverse. Let $x \in \mathfrak{D}(T)$, $y \in \mathcal{H}$ such that $\iota\iota^* y \in \mathfrak{D}(T^*T)$. We see that

$$\begin{aligned} \langle \iota x, y \rangle_{\mathcal{H}} = \langle x, \iota^* y \rangle_T &= \langle \iota x, \iota\iota^* y \rangle_{\mathcal{H}} + \langle T\iota x, T\iota\iota^* y \rangle_{\mathcal{H}} \\ &= \langle \iota x, \iota\iota^* y \rangle_{\mathcal{H}} + \langle \iota x, T^*T\iota\iota^* y \rangle_{\mathcal{H}} = \langle \iota x, \iota\iota^* y + T^*T\iota\iota^* y \rangle_{\mathcal{H}} \,. \end{aligned}$$

Substracting the left hand side, we get

$$0 = \langle \iota x, \iota\iota^* y + T^*T\iota\iota^* y - y \rangle .$$

This equality holds for all $x \in \mathfrak{D}(T)$, giving

$$y = \iota\iota^* y + T^*T\iota\iota^* y = (1 + T^*T)\iota\iota^* y,$$

which implies that $\iota\iota^* y = (I + T^*T)^{-1} y$, for all $y \in (\iota\iota^*)^{-1}(\mathfrak{D}(T^*T))$. Now we need to show that $(\iota\iota^*)^{-1}(\mathfrak{D}(T^*T))$ is dense in $\mathcal{H}$. To do that, let $x \in \mathfrak{D}(T)$, $y \in \mathcal{H}$. Then

$$\begin{aligned} \langle T\iota x, T\iota\iota^* y\rangle_{\mathcal{H}} &= \langle x, \iota^* y\rangle_T - \langle \iota x, \iota\iota^* y\rangle_{\mathcal{H}} \\ &= \langle \iota x, y\rangle_{\mathcal{H}} - \langle \iota x, \iota\iota^* y\rangle_{\mathcal{H}} = \langle \iota x, y - \iota\iota^* y\rangle_{\mathcal{H}} . \end{aligned}$$

This shows that $T^*(T\iota\iota^* y) = y - \iota\iota^* y$, proofing that $\iota\iota^* y \in \mathfrak{D}(T^*T)$. As $y \in \mathcal{H}$ was arbitrary, this gives $(\iota\iota^*)^{-1}\mathfrak{D}(T^*T) = \mathcal{H}$, completing the proof that $\iota\iota^* = (1 + T^*T)^{-1}$. □

Define $A := \iota\iota^*$ and $B := TA$. If we think of A corresponding to $1/1+|x|^2$, then we would expect B to be bounded as well. As it turns out, this is true.

Lemma 3.3. *$B = TA = T(I + T^*T)^{-1}$ is a bounded operator, and we have $AT \subset TA$.*

Proof. Let $x \in \mathfrak{D}(I + T^*T)$ such that $(I + T^*T)x = y \in \mathfrak{D}(T)$. Using the calculation at the end of Lemma 3.1, we get $\|y + T^*Ty\|^2 \geq \|Ty\|^2$. From that, $\|TAy\|^2 = \|Tx\|^2 \leq \|(I + T^*T)x\|^2 = \|y\|^2$, which proves that B is bounded.

To show that $AT \subset TA$, take $y \in \mathfrak{D}(AT) = \mathfrak{D}(T)$, $x \in \mathfrak{D}(T^*T)$ such that $y = (I + T^*T)x$. $T^*Tx \in \mathfrak{D}(T)$ which implies $Tx \in \mathfrak{D}(TT^*) = \mathfrak{D}(T^*T)$. Then

$$ATy = A(Tx + TT^*Tx) = A(I + T^*T)Tx = Tx,$$

and

$$TAy = T(I + T^*T)^{-1}(I + T^*T)x = Tx,$$

concluding that $AT = TA$ on $\mathfrak{D}(T)$.

□

The operator AT is bounded but not defined on all of $\mathcal{H}$. So we extend it in the following

Lemma 3.4. *AT admits a bounded linear extension $\overline{AT}$ to all of $\mathcal{H}$. We then have $\overline{AT} = TA$.*

Proof. For $x \in \mathcal{H}$ we can choose a sequence $x_n \in \mathfrak{D}(T)$, with $x_n \to x$. Define $\overline{AT}(x) := TA(x)$. This is linear, because $AT = TA$ on $\mathfrak{D}(T)$. As TA is bounded, the limit does not depend on the chosen sequence. □

Remark 3.5. The previous two lemmata and their proofs, still hold if we replace T by T^*, giving us

$$AT^* = T^*A \text{ and hence } B^* = T^*A.$$

One also has the identity

$$\begin{aligned}A^2 + B^*B &= (I + T^*T)^{-2} + T^*(I + T^*T)^{-1}T(I + T^*T)^{-1} \\ &= (I + T^*T)^{-2} + T^*T(I + T^*T)^{-1}(I + T^*T)^{-1} \\ &= (I + T^*T)(I + T^*T)^{-2} \\ &= (I + T^*T)^{-1} \\ &= A.\end{aligned}$$

From now on, we identify $B = AT$ and $B^* = AT^*$ with their bounded extensions. Define $\mathfrak{A} = \mathfrak{A}(T)$ by $\mathfrak{A} := \langle I, A, B\rangle$. Let $\chi \in \mathrm{Sp}(\mathfrak{A})$, such that $\chi(A) = 0$. By the previous identity, we get

$$\chi(A)^2 + |\chi(B)|^2 = \chi(A),$$

which implies that $\chi(B) = 0$ as well. But for all χ in $\mathrm{Sp}(\mathfrak{A})$, it holds that $\chi(I) = 1$. If such a χ exists, it is therefore unique. We call this character χ_∞.

Define $\theta\colon \mathrm{Sp}(\mathfrak{A}) \to \overline{\mathbb{C}}$ by

$$\chi \mapsto \begin{cases} \frac{\chi(B)}{\chi(A)} & \text{, if } \chi \neq \chi_\infty \\ \infty & \text{, if } \chi = \chi_\infty. \end{cases}$$

Let $\chi \neq \chi_\infty$. Since χ is a involutive algebra homomorphism, $A^2 + B^*B = A$ implies for $\chi(A) = \chi(A)\overline{\chi(A)} + \chi(B)\overline{\chi(B)}$, which is equivalent to $\frac{1}{\chi(A)} = 1 + \frac{\chi(B)}{\chi(A)}\overline{\left(\frac{\chi(B)}{\chi(A)}\right)} = 1 + |\theta(\chi)|^2$. The last equalities hold because A is self-adjoint, implying that $\chi(A)$ is a real number. Inverting the last equality gives

$$\chi(A) = \frac{1}{1 + |\theta(\chi)|^2}. \tag{$*$}$$

The definition of θ (and not $T =$ "$\frac{B}{A}$"), gives

$$\chi(B) = \chi(A)\frac{\chi(B)}{\chi(A)} = \chi(A)\ \theta(\chi). \tag{$**$}$$

Recalling the definition of the Gelfand transform, we see that our map

$$\theta\colon \mathrm{Sp}(\mathfrak{A}) \setminus \{\chi_\infty\} \to \mathbb{C}$$

equals a fraction of two single Gelfand transform

$$\theta(\chi) = \frac{\chi(B)}{\chi(A)} = \frac{\mathcal{G}_B(\chi)}{\mathcal{G}_A(\chi)} = \frac{\mathcal{G}_B}{\mathcal{G}_A}(\chi).$$

But $\mathcal{G}_A(\chi) \neq 0$ on $\mathrm{Sp}(\mathfrak{A}) \setminus \{\chi_\infty\}$, which implies that θ is continuous on $\mathrm{Sp}(\mathfrak{A}) \setminus \{\chi_\infty\}$. To show that θ is continuous at χ_∞ , let $(\chi_\lambda)_{\lambda\in\Lambda}$ be a net converging to χ_∞ and $\chi_\lambda \neq \chi_\infty$ for all $\lambda \in \Lambda$. If no such net exists, we need not worry about continuity as χ_∞ does not even exist, or is an isolated point in $\mathrm{Sp}(\mathfrak{A})$. By continuity of $\mathcal{G}_A$ we have

$$\mathcal{G}_A(\chi_\lambda) = \chi_\lambda(A) \to \chi_\infty(A) = 0.$$

Equation $(*)$ implies

$$|\theta(\chi_\lambda)|^2 + 1 = \frac{1}{\chi(A)} \to \infty,$$

which is equivalent to

$$|\theta(\chi_\lambda)| \to \infty.$$

This implies that θ is continuous. Summarizing the previous pragraph, we get

Lemma 3.6. *θ extends to a continuous map $\theta \colon \operatorname{Sp}(\mathfrak{A}) \to \overline{\mathbb{C}}$, by $\theta(\chi_\infty) := \infty$. Furthermore $\theta \colon \operatorname{Sp}(\mathfrak{A}) \to \overline{\mathbb{C}}$ is a homeomorphism onto its image.*

Proof. The first claim was proven before. For the second claim, we check that θ is injective: Let $\chi_1, \chi_2 \neq \chi_\infty$. Equations $(*)$ and $(**)$ imply that, if $\theta(\chi_1) = \theta(\chi_2)$, χ_1 coincides with χ_2. Furthermore, χ_∞ is unique, which implies that θ is injective. Since $\operatorname{Sp}(\mathfrak{A})$ is compact and $\overline{\mathbb{C}}$ is Hausdorff, this proves the second claim. □

Combining the Gelfandisomorphism $\mathcal{G} \colon \mathfrak{A} \to \mathrm{C}(\operatorname{Sp}\mathfrak{A})$, with θ, one has

$$\begin{gathered} \mathfrak{A} \longrightarrow \mathrm{C}(\operatorname{Sp}\mathfrak{A}) \longrightarrow \mathrm{C}(\theta(\operatorname{Sp}\mathfrak{A})) \\ x \mapsto \mathcal{G}_x \ ; \ f \mapsto f \circ \theta^{-1} \\ \mathcal{G}^{-1} f \leftarrow\!\shortmid f \ ; \ g \circ \theta \leftarrow\!\shortmid g. \end{gathered}$$

Define

$$\Phi \colon \mathrm{C}(\theta(\operatorname{Sp}\mathfrak{A})) \to \mathfrak{A}, \ g \mapsto \mathcal{G}^{-1}(g \circ \theta).$$

As $\mathcal{G}$ is an involutive algebra homomorphism, Φ is as well.

Proposition 3.7. *Let T be a normal operator on $\mathcal{H}$, $\mathfrak{A} = \langle I, A, B \rangle$ the C^*-algebra associated to T. Then Φ is the only involutive algebra homomorphism from $\mathrm{C}(\theta(\operatorname{Sp}\mathfrak{A}))$ onto $\mathfrak{A}$, such that*

$$\Phi\left(\frac{1}{1+|z|^2}\right) = A, \ \Phi\left(\frac{z}{1+|z|^2}\right) = B.$$

Proof. Let $z \in \theta(\operatorname{Sp}\mathfrak{A})$, $z = \theta(\chi)$. Using $\mathcal{G}_A(\chi) = \chi(A) = \frac{1}{1+|\theta(\chi)|^2}$, we get

$$\Phi^{-1}(A)(z) = (\mathcal{G}_A \circ \theta^{-1})(z) = \mathcal{G}_A(\theta^{-1}(z)) = \frac{1}{1+|z|^2}.$$

Moreover, using $\mathcal{G}_B(\chi) = \chi(A)\theta(\chi) = \frac{\theta(\chi)}{1+|\theta(\chi)|^2}$, we get

$$\Phi^{-1}(B)(z) = (\mathcal{G}_B \circ \theta^{-1})(z) = \mathcal{G}_B(\theta^{-1}(z)) = \frac{z}{1+|z|^2}.$$

To proof uniqueness, let $\Psi \colon \mathrm{C}(\theta(\operatorname{Sp}\mathfrak{A})) \to \mathfrak{A}$ be another involutive algebra homomorphism, such that

$$\Psi\left(\frac{1}{1+|z|^2}\right) = A, \text{ and } \Psi\left(\frac{z}{1+|z|^2}\right) = B.$$

Then Φ coincides with Ψ on all polynomials in $A, B, \overline{B}$. Theses polynomials form an involutive algebra, which seperates points. By the theorem of Stone-Weierstrass, it is dense, and therefore Φ equals Ψ as they are continious. □

Our goal is to construct a functional calculus for T. With respect to Φ, T would corresponds to $\mathrm{id}_{\theta(\mathrm{Sp}\,\mathfrak{A})}$. But if $\chi_\infty \in \mathrm{Sp}(\mathfrak{A})$, then $\mathrm{id}_{\theta(\mathrm{Sp}\,\mathfrak{A}))} \notin \mathrm{C}(\mathrm{Sp}\,\mathfrak{A})$ since $\infty \notin \mathbb{C}$.

4 Extension of Continuous Spectral Measures

This section will follow [4] and [6, pp. 341- 345] , while some notation is taken from [5, Ch. 6]. The reader not familiar with complex measures, is advised to review the basic facts and definitions in [7, Ch. 6].

We want to extend a given spectral measure for continuous functions, like the one we obtained in Proposition 2.3, to measurable ones. To do so, we need to check whether the operators we attain are bounded, or densely defined.

Definition 4.1 (Spectral measure)**.** Let X be a compact space and let

$$\Phi\colon \mathrm{C}(X) \to \mathcal{B}(\mathcal{H})$$

be a map. Φ is called a *spectral measure*, if its image $\mathfrak{A} := \Phi(\mathrm{C}(X))$ is a commutative C^*-algebra of $\mathcal{B}(\mathcal{H})$ and Φ induces an isomorphism onto its image.

Remark 4.2. By isomorphism we mean that

1. Φ is an involutive algebra homomorphism,
2. Φ is a bijection onto $\mathfrak{A}$,
3. Φ is an isometry: $\|\Phi f\| = \|f\|_\infty$.

Example 4.3. The maps constructed in Propositions 3.7 and 2.3 are examples of spectral measures.

Let $\mathfrak{m}$ be a positive Radon integral on a compact space X, meaning a continuous linear functional on $\mathrm{C}(X)$, such that $\mathfrak{m}(f) = 0$ implies that $f = 0$. A more in-depth study of Radon integrals can be found in [5, Chapter 6.1]. Set $\mathcal{H} := L^2(\mathfrak{m})$. We define

$$\begin{aligned}\Phi\colon \mathrm{C}(X) &\to \mathcal{B}(\mathrm{L}^2\mathfrak{m})\\ f &\mapsto (g \mapsto f \cdot g).\end{aligned}$$

Φ is a spectral measure. This claim is not obvious, and the proof that Φ is an isometry requires a bit of work, and is omitted here. It can be found in [4]. This example is in a certain sense the most general example of a spectral measure, shown in Theorem 6.4.

Departing from this example, let $\mathcal{H}$ be a Hilbert space and $\Phi\colon \mathrm{C}(X) \to \mathcal{B}(\mathcal{H})$ be a spectral measure. For all $g, h \in \mathcal{H}$ define

$$\mathfrak{m}_{g,h}(f) := \langle g, \Phi_f h \rangle \,.$$

We see, that the map $f \mapsto \mathfrak{m}_{g,h}(f)$ is a linear form on $\mathrm{C}(X)$ for every pair $(g, h) \in \mathcal{H} \times \mathcal{H}$.

Theorem 4.4. *For all $g, h, k \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and $f, \phi, \psi \in \mathrm{C}(X)$, it holds that $\mathfrak{m}_{g,h}$ is a Radon integral on X with the following properties:*

(i) $\|\mathfrak{m}_{g,h}\| \leq \|g\|\|h\|$

(ii) $\mathfrak{m}_{\alpha g+h,\beta k} = \overline{\alpha}\beta\mathfrak{m}_{g,k} + \beta\mathfrak{m}_{h,k}$

(iii) $\overline{\mathfrak{m}}_{g,h} = \mathfrak{m}_{h,g}$, $\overline{\mathfrak{m}}_{g,h}\colon f \mapsto \overline{\mathfrak{m}}_{g,h}(f) = \overline{\mathfrak{m}_{g,h}(\overline{f})}$

(iv) $\mathfrak{m}_{g,g} \geq 0$

(v) $\mathfrak{m}_{\Phi_\phi g, \Phi_\psi h} = \overline{\phi}\psi \mathfrak{m}_{g,h}$.

Proof. (i)

$$\begin{aligned}\|\mathfrak{m}_{g,h}\| &= \sup_{\|\phi\|_\infty \leq 1} |\mathfrak{m}_{g,h}(\phi)| \\ &= \sup_{\|\phi\|_\infty \leq 1} |\langle g, \Phi_\phi h\rangle| \\ &\leq \sup_{\|\phi\|_\infty \leq 1} \|g\|\|\Phi_\phi\|\|h\| \\ &= \|g\|\|h\|.\end{aligned}$$

The last equality holds, because Φ is an isometry.

(ii) follows immediately from the linearity of $\langle \cdot, \cdot \rangle$.

(iii) $\overline{\mathfrak{m}}_{g,h}(f) = \overline{\left\langle g, \Phi_{\overline{f}} h \right\rangle} = \left\langle \Phi_{\overline{f}} h, g \right\rangle = \langle h, \Phi_f g\rangle = \mathfrak{m}_{h,g}(f)$.

(iv) Let $\phi \geq 0$. Because Φ is algebra homomorphism, and $\overline{\sqrt{\phi}} = \sqrt{\phi}$, we have $\Phi_\phi = \Phi_{\sqrt{\phi}}\Phi_{\sqrt{\phi}} = \Phi^*_{\sqrt{\phi}}\Phi_{\sqrt{\phi}}$. Therefore

$$\mathfrak{m}_{g,g}(\phi) = \langle g, \Phi_\phi g\rangle = \left\langle \Phi_{\sqrt{\phi}} g, \Phi_{\sqrt{\phi}} g\right\rangle \geq 0.$$

(v) $\mathfrak{m}_{\Phi_\phi g, \Phi_\psi h}(f) = \langle \Phi_\phi g, \Phi_f \Phi_\psi h\rangle = \left\langle g, \Phi_{\overline{\phi}\psi f} h\right\rangle = (\overline{\phi}\psi\mathfrak{m}_{g,h})(f)$.

□

We now extend $\mathfrak{m}_{g,h}$ to measurable functions in the sense of [5, Ch. 4.5]. This extension is unique, as the measurable (or Borel) functions are the monotone sequential completion of the continuous functions [5, Propostition 6.2.9], and the theorem of monotone convergence holds [5, Theorem 6.1.13]. The definition of measurability without using σ-algebras, can be found in [5, Ch. 6.3].

Definition 4.5. $N \subset X$ is called a Φ-set of measure zero, or Φ-null set, if N is a null set of $|\mathfrak{m}_{g,h}|$ for all $g, h \in \mathcal{H}$, that is $|\mathfrak{m}_{g,h}|(N) = 0$.

Note that $|\mathfrak{m}_{g,h}|$ is a real valued Radon integral. A function $f\colon X \to \mathbb{C}$ is called Φ-measurable, if f is $|\mathfrak{m}_{g,h}|$-measurable for all $g, h \in \mathcal{H}$. Denote the set of all measurable functions $\mathrm{L}^0(\Phi)$.

$$\begin{aligned}\mathrm{L}^1(\Phi) &:= \left\{ f \in \mathrm{L}^0(\Phi) \;\middle|\; f \in \mathrm{L}^1(\mathfrak{m}_{g,h}) \text{ for all } g, h \in \mathcal{H}\right\}, \\ \mathrm{L}^\infty(\Phi) &:= \left\{ f \in \mathrm{L}^0(\Phi) \;\middle|\; \|f\|_\infty < \infty \right\}\end{aligned}$$

where,

$$\|f\|_\infty := \inf\left\{\lambda > 0 \;\middle|\; |f| \leq \lambda,\ \Phi\text{-a.e.}\right\}.$$

By $\mathscr{E}(X)$ we denote the measurable subsets of X:

$$\mathscr{E}(X) := \left\{ A \subset X \;\middle|\; 1_A \in \mathrm{L}^0(\Phi)\right\}.$$

For $f \in \mathrm{L}^0(\Phi)$, we set

$$\begin{aligned}\mathfrak{D}(f) &:= \Big\{h \in \mathcal{H} \;\Big|\; f \in \mathrm{L}^1(\mathfrak{m}_{g,h}) \text{ for all } g \in \mathcal{H} \\ &\qquad\qquad \text{and } g \mapsto \int f \,\mathrm{d}\mathfrak{m}_{g,h} \text{ is continuous}\Big\} \\ &= \Big\{h \in \mathcal{H} \;\Big|\; f \in \mathrm{L}^1(\mathfrak{m}_{g,h}) \text{ for all } g \in \mathcal{H} \\ &\qquad\qquad \text{and } \exists k \in \mathcal{H} \text{ such that } \int f \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, k\rangle \Big\},\end{aligned}$$

where the second equality is due to the Riesz representation theorem. The reader familiar with unbounded operators, will recognize the similarity with the definition of the adjoint operator. Using the second equality, we define

$$\Phi_f h := k(h, f) = k \text{ for } h \in \mathfrak{D}(f) =: \mathfrak{D}(\Phi_f).$$

In other words we have

$$\int f \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_f h\rangle = \langle g, k\rangle .$$

By sesquilinearity of $\langle \cdot, \cdot \rangle$, Φ_f is linear as well.

Remark 4.6. If we want to proof claims about $\mathfrak{m}_{g,h}$ acting on $\mathrm{L}^0(\Phi)$ it is enough to show them for $\mathrm{C}(X)$, as every measurable function is the limit of continuous ones [5, Proposition 6.2.9].

Lemma 4.7. *For all $f \in L^0(\Phi)$, $h \in \mathfrak{D}(f)$, and $g \in \mathcal{H}$ it holds that*

$$\mathfrak{m}_{g,\Phi_f h} = f\mathfrak{m}_{g,h} \qquad \textit{and} \qquad \mathfrak{m}_{\Phi_f g,h} = \overline{f}\mathfrak{m}_{g,h}.$$

Proof. By the remark, let $\phi \in \mathrm{C}(X)$. Since Φ is a spectral measure, we get $\Phi^*_\phi = \Phi_{\overline{\phi}}$. Thus,

$$\begin{aligned}\mathfrak{m}_{g,\Phi_f h}(\phi) &= \langle g, \Phi_\phi(\Phi_f h)\rangle = \left\langle \Phi_{\overline{\phi}} g, \Phi_f h \right\rangle = \int f \,\mathrm{d}\mathfrak{m}_{\Phi_{\overline{\phi}} g,h} \\ &= \int f \,\mathrm{d}\mathfrak{m}_{g,\Phi_\phi h} = \int f\phi \,\mathrm{d}\mathfrak{m}_{g,h} = (f\mathfrak{m}_{g,h})(\phi).\end{aligned}$$

Since h being in the domain of f means, that for all $g \in \mathcal{H}$, f is an element of $\mathrm{L}^1(\mathfrak{m}_{g,h})$, the expression $f\mathfrak{m}_{g,h}$ makes sense.

On the other hand, let $g \in \mathfrak{D}(f)$. We calculate

$$\begin{aligned}\mathfrak{m}_{\Phi_f g,h}(\phi) &= \langle \Phi_f g, \Phi_\phi h\rangle = \overline{\langle \Phi_\phi h, \Phi_f g\rangle} = \overline{\left\langle h, \Phi_{\overline{\phi}} \Phi_f g \right\rangle} \\ &= \overline{\int \overline{\phi} \,\mathrm{d}\mathfrak{m}_{h,\Phi_f g}} = \overline{\int \overline{\phi} f \,\mathrm{d}\mathfrak{m}_{h,g}} = \overline{\mathfrak{m}_{h,g}(\overline{\phi} f)} \\ &= \overline{\mathfrak{m}}_{h,g}(\phi \overline{f}) = \mathfrak{m}_{g,h}(\phi \overline{f}) = (\overline{f}\mathfrak{m}_{g,h})(\phi).\end{aligned}$$

□

We now want to know, whether the operators we obtain from our expanded measure are bounded, or at least densely defined. This will be done in the next lemmata, which are going to be summarized in a theorem at the end of the chapter.

Lemma 4.8. *If $f \in L^\infty(\Phi)$, then $\mathfrak{D}(f) = \mathcal{H}$, $\Phi_f \in \mathscr{L}(\mathcal{H})$ and*

$$\|\Phi_f\| \leq \|f\|_\infty = \inf\left\{\alpha > 0 \;\middle|\; \alpha \geq |f|,\ \Phi\text{-}a.e.\right\}.$$

Proof. Let $f \in \mathrm{L}^\infty(\Phi)$. We have to show that $\mathfrak{D}(f) = \mathcal{H}$. Let $g, h \in \mathcal{H}$. One gets:

$$\begin{aligned}\left|\int f \,\mathrm{d}\mathfrak{m}_{g,h}\right| &\leq \int |f|\mathrm{d}|\mathfrak{m}_{g,h}| \\ &\leq \|f\|_\infty \int \mathrm{d}|\mathfrak{m}_{g,h}| \\ &= \|f\|_\infty \|\mathfrak{m}_{g,h}\| \\ &\leq \|f\|_\infty \|h\|\|g\|\end{aligned}$$

which implies

$$g \mapsto \int f \,\mathrm{d}\mathfrak{m}_{g,h}$$

is continuous and f an element of $\mathscr{L}^1(\mathfrak{m}_{g,h})$ for all $g, h \in \mathcal{H}$. The third equality follows from

$$\int \mathrm{d}|\mathfrak{m}_{g,h}| = |\mathfrak{m}_{g,h}|(1) = \sup_{\substack{|\phi| \leq 1 \\ \phi \in \mathrm{C}(X)}} |\mathfrak{m}_{g,h}|(\phi) = \sup_{\|\phi\|_\infty \leq 1} |\mathfrak{m}_{g,h}|(\phi) = \|\mathfrak{m}_{g,h}\|.$$

This shows that the assignment $g \mapsto \int f \,\mathrm{d}\mathfrak{m}_{g,h}$ defines a continuous map.

$$\|\Phi_f\| = \sup_{\|g\|,\|h\| \leq 1} |\langle g, \Phi_f h\rangle| = \sup_{\|g\|,\|h\| \leq 1} \left|\int f \,\mathrm{d}\mathfrak{m}_{g,h}\right| \leq \|f\|_\infty. \qquad (*)$$

The other inequality is also true and will be proven in Lemma 4.14. □

Remark 4.9. Because of $(*)$ convergence of f_n in $\mathrm{L}^\infty(\Phi)$, implies convergence of Φ_{f_n} in $\mathcal{B}(\mathcal{H})$.

By $\int^* f \,\mathrm{d}\mathfrak{m}_{g,h}$, we mean the upper integral, as defined in [5, Ch. 6.1]. If $\int^* f \,\mathrm{d}\mathfrak{m}_{g,h} < \infty$, then $\int f \,\mathrm{d}\mathfrak{m}_{g,h} = \int^* f \,\mathrm{d}\mathfrak{m}_{g,h}$, also found in [5].

Lemma 4.10. *Let $f \in L^0(\Phi), (f_n)$ a net in $L^\infty(\Phi)$, and $\alpha, \beta \geq 0$ such that*

$$\begin{aligned}|f_n| &\leq \alpha|f| + \beta \text{ for all } n \\ f_n &\to f\ \ \Phi\text{-}a.e..\end{aligned}$$

Then the following statements about $h \in \mathcal{H}$ are equivalent:

(i) $h \in \mathfrak{D}(f)$

(ii) $\int^* |f|^2 \,\mathrm{d}\mathfrak{m}_{h,h} < \infty$

(iii) $(\Phi_{f_n} h)$ *converges in* $\mathcal{H}$.

One then has

$$\|\Phi_f h\|^2 = \int |f|^2 \,\mathrm{d}\mathfrak{m}_{h,h} \text{ and}$$

$$\Phi_f h = \lim \Phi_{f_n} h.$$

For example, one can take the net $f_n = 1_{\mathcal{A}_n}$, where $\mathcal{A}_n = \left\{x \in X \;\middle|\; |f(x)| \leq n\right\}$.

Proof. *(i)* $\Rightarrow$ *(ii)*: Let $h \in \mathfrak{D}(f)$.

$$\infty > \|\Phi_f h\|^2 = \langle \Phi_f h, \Phi_f h \rangle = \int^* f \, \mathrm{d}\mathfrak{m}_{\Phi_f h, h} = \int^* f\overline{f} \, \mathrm{d}\mathfrak{m}_{h,h} = \int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h}.$$

(ii) $\Rightarrow$ *(iii)*: Let $h \in \mathscr{L}^2(\mathfrak{m}_{h,h})$. It is enough to show that Φ_{f_n} is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. For $g \in \mathcal{H}$

$$\begin{aligned} \langle g, (\Phi_{f_m} - \Phi_{f_n})h \rangle &= \langle g, \Phi_{f_m} \rangle h - \langle g, \Phi_{f_n} h \rangle \\ &= \int f_m \, \mathrm{d}\mathfrak{m}_{g,h} - \int f_n \, \mathrm{d}\mathfrak{m}_{g,h} \\ &= \int (f_m - f_n) \, \mathrm{d}\mathfrak{m}_{g,h} \\ &= \left\langle g, \Phi_{(f_m - f_n)} h \right\rangle. \end{aligned}$$

Since g was arbitrary, we get $(\Phi_{f_m} - \Phi_{f_n})h = \Phi_{(f_m - f_n)}h$, and

$$\begin{aligned} \|(\Phi_{f_m} - \Phi_{f_n})h\|^2 &= \|\Phi_{(f_m - f_n)}h\|^2 \\ &= \int |f_m - f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h}. \end{aligned}$$

By our assumption, we have $f_m - f_n \to 0$ Φ-a.e. Therefore

$$|f_m - f_n|^2 \leq (2(\alpha|f| + \beta))^2 \in \mathscr{L}^1(\mathfrak{m}_{h,h}),$$

as constant functions are contained $\mathrm{L}^1(\Phi)$. Now, Lebegues theorem about dominated convergence yields the claim.

(iii) $\Rightarrow$ *(ii)*: Using Fatou's lemma, and $\|\Phi_{f_n} h\|^2 = \int |f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h}$, we get

$$\begin{aligned} \infty > \|\lim_{n\to\infty} \Phi_{f_n} h\|^2 &= \lim_{n\to\infty} \|\Phi_{f_n} h\|^2 = \lim_{n\to\infty} \int^* |f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h} \\ &\geq \int^* \liminf_{n\to\infty} |f_n|^2 \, \mathrm{d}\mathfrak{m}_{h,h} = \int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h}. \end{aligned}$$

(ii) $\Rightarrow$ *(i)*: By the theorem of Radon–Nikodym, there exists a Borel measurable function $\phi \colon X \to \mathbb{C}$, $|\phi| = 1$, such that

$$|\mathfrak{m}_{g,h}| = \phi \mathfrak{m}_{g,h}.$$

Now define $\widetilde{f} \coloneqq \phi|f|$, $\widetilde{f_n} \coloneqq \phi|f_n|$. Thus

$$\int^* |\widetilde{f}|^2 \, \mathrm{d}\mathfrak{m}_{h,h} = \int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} < \infty.$$

As shown in *(ii)* $\Rightarrow$ *(iii)*, the limit of $\Phi_{\widetilde{f_n}}$ exists. Hence, for $g \in \mathcal{H}$

$$\begin{aligned} \infty > \|g\|^2 \|\lim_{n\to\infty} \Phi_{\widetilde{f_n}} h\|^2 &\geq \left| \left\langle g, \lim_{n\to\infty} \Phi_{\widetilde{f_n}} h \right\rangle \right| = \left| \lim_{n\to\infty} \int \widetilde{f_n} \, \mathrm{d}\mathfrak{m}_{g,h} \right| \\ = \lim_{n\to\infty} \int |f_n| \mathrm{d}|\mathfrak{m}_{g,h}| &\geq \int^* \liminf_{n\to\infty} |f_n| \mathrm{d}|\mathfrak{m}_{g,h}| = \int^* |f| \mathrm{d}|\mathfrak{m}_{g,h}|, \end{aligned}$$

which implies that $f \in \mathscr{L}^1(\mathfrak{m}_{g,h})$. So once more by Lebegues theorem

$$\left\langle g, \lim_{n\to\infty} \Phi_{f_n} h \right\rangle = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mathfrak{m}_{g,h} = \int f \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_f h \rangle ,$$

proving that, $h \in \mathfrak{D}(f)$.

□

Lemma 4.11. *For each $f \in L^0(\Phi)$, Φ_f is a normal Operator, and we have*

$$\Phi_f^* = \Phi_{\overline{f}}.$$

If f is real valued, then Φ_f is self-adjoint, and $\Phi_{1_A} =: E_A$ for $A \in \mathscr{E}(X)$ is an orthogonal projection.

Proof. First we show that Φ_f is densely defined. We claim $E_{\mathcal{A}_n}(\mathcal{H}) \subset \mathfrak{D}(f)$, where $\mathcal{A}_n = \left\{x \in X \;\middle|\; |f(x)| \leq n\right\}$. Let $h \in \mathcal{H}$. By Lemma 4.10 the claim is equivalent to

$$\int^* |f|^2 \,\mathrm{d}\mathfrak{m}_{E_{\mathcal{A}_n} h, E_{\mathcal{A}_n} h} < \infty.$$

We have

$$\int^* |f|^2 \,\mathrm{d}\mathfrak{m}_{E_{\mathcal{A}_n} h, E_{\mathcal{A}_n} h} = \int^* \overline{1}_{\mathcal{A}_n} 1_{\mathcal{A}_n} |f|^2 \,\mathrm{d}\mathfrak{m}_{h,h} \leq n^2 \int \mathrm{d}\mathfrak{m}_{h,h} \leq n^2 \|h\|^2 < \infty.$$

For $h \in \mathcal{H}$, we have $h = \lim_{n\to\infty} E_{\mathcal{A}_n} h$, as $1_{\mathcal{A}_n} \to 1$ pointwise Φ-a.e. Let $h \in \mathcal{H} = \mathfrak{D}(1)$. By Lemma 4.10 we have

$$h = \Phi_1 h = \lim_{n\to\infty} \Phi_{1_{\mathcal{A}_n}} h = \lim_{n\to\infty} E_{\mathcal{A}_n} h,$$

which gives $\mathfrak{D}(f) \subset \mathcal{H}$ is dense, as $E_{\mathcal{A}_n} h \in \mathfrak{D}(f)$ by the claim.

Now, we claim $\Phi_{\overline{f}} \subset \Phi_f^*$. Let $g, h \in \mathfrak{D}(f) = \mathfrak{D}(\overline{f})$. Using Theorem 4.4 (*iii*), one has

$$\langle g, \Phi_f h \rangle = \int f \,\mathrm{d}\mathfrak{m}_{g,h} = \overline{\int \overline{f} \mathrm{d}\overline{\mathfrak{m}_{g,h}}} = \overline{\left\langle h, \Phi_{\overline{f}} g \right\rangle} = \left\langle \Phi_{\overline{f}} g, h \right\rangle .$$

Thus, $g \in \mathfrak{D}(\Phi_f^*)$ and $\Phi_f^* g = \Phi_{\overline{f}} g$ for all $g \in \mathfrak{D}(f)$.

On the other hand, to show that $\Phi_{\overline{f}} \supset \Phi_f^*$, let $g \in \mathfrak{D}(\Phi_f^*)$. By Lemma 4.10, we only have to show $\Phi_{\overline{f}_n} g$ converges in $\mathcal{H}$, for some net satisfyig the conditions of Lemma 4.10. As a net, we take $f_n = f \cdot 1_{\mathcal{A}_n}$, where $\mathcal{A}_n = \left\{x \in X \;\middle|\; |f(x)| \leq n\right\}$. Let $h \in \mathcal{H}$. For better readability, we write E_n for $\Phi_{\mathcal{A}_n}$, F_n, F_n^* for Φ_{f_n} respectively $\Phi_{\overline{f}_n}$, and F for Φ_f. Note that F^* stands for Φ_F^* not for $\Phi_{\overline{f}}$.

$$\begin{aligned}
\left\langle \Phi_{\overline{f}_n} g, h \right\rangle &= \langle F_n^* g, h \rangle = \int \mathrm{d}\mathfrak{m}_{F_n^* g, h} = \int f_n \,\mathrm{d}\mathfrak{m}_{g,h} = \int f \,\mathrm{d}\mathfrak{m}_{g, E_n h} \\
&= \int \mathrm{d}\mathfrak{m}_{g, F E_n h} = \langle g, F E_n h \rangle = \langle F^* g, E_n h \rangle = \int 1_{\mathcal{A}_n} \,\mathrm{d}\mathfrak{m}_{F^* g, h} \\
&= \int \mathrm{d}\mathfrak{m}_{E_n F^* g, h} = \left\langle E_{\mathcal{A}_n} \Phi_{\overline{f}} g, h \right\rangle
\end{aligned}$$

Now $\Phi_{\overline{f}_n} g = E_{\mathcal{A}_n} \Phi^*_f g \xrightarrow{n\to\infty} \Phi^*_f g$, as $E_{\mathcal{A}_n}$ converges to the identity. It follows, that $\Phi_{\overline{f}} \supset \Phi^*_f$, completing the proof of $\Phi_{\overline{f}} = \Phi^*_f$.

Next, we claim Φ_f is a normal element. We have to show, that $\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi^*_f)$ and $\|\Phi^*_f h\| = \|\Phi_f h\|$. The first claim is already proven, because $\mathfrak{D}(f) = \mathfrak{D}(\overline{f})$. The second condition is also fulfilled, which can be seen by the norm formula of Lemma 4.10. Thus $\mathfrak{D}(\Phi^*_f) = \mathfrak{D}(\Phi_{\overline{f}})$ and $\|\Phi^*_f h\| = \|\Phi_f h\|$, which proves that Φ_f is normal. To show that this implies the usual definition of a normal operator, i.e. $\Phi_f \Phi^*_f = \Phi^*_f \Phi_f$, can easily be seen, using the polarization identity, found in [9, Ch. 4.6].

If f is real valued, we have that $f = \overline{f}$, which gives the selfadjointness of Φ_f. Furthermore $E^*_{\mathcal{A}} = E_{\mathcal{A}}$. As $(E_{\mathcal{A}})^2 = E_{\mathcal{A}}$, we get that $E_{\mathcal{A}}$ is an orthogonal projection.

□

Corollary 4.12.

1. $f \in L^\infty(\Phi)$,
$$f \geq 0 \ \Phi\text{-}a.e. \Rightarrow \Phi_f \geq 0$$
2. $\mathcal{A} \in \mathscr{E}(X)$,
$$E_{\mathcal{A}} = 0 \Leftrightarrow \mathcal{A} \ \Phi\text{-}null\ set$$
3. $U \in \mathscr{E}(X)$ *open,*
$$U \neq \varnothing \Rightarrow E_U \neq 0$$

Proof. Immediate consequence of the previous lemmata. Full proof found in [4]. □

Lemma 4.13. *For each* $\varphi, \psi \in L^0(\Phi)$, $\alpha \in \mathbb{C}$, *we have*

(i) $\Phi_{\alpha\varphi} = \alpha \Phi_\varphi$

(ii) $\mathfrak{D}(\Phi_\varphi \Phi_\psi) = \mathfrak{D}(\varphi\psi) \cap \mathfrak{D}(\psi)$, *and* $\overline{\Phi_\varphi \Phi_\psi} = \Phi_{\varphi\psi}$

(iii) $\overline{\Phi_\varphi + \Phi_\psi} = \Phi_{\varphi+\psi}$

(iv) $\psi \in L^\infty(\Phi) \Rightarrow \Phi_\varphi + \Phi_\psi = \Phi_{\varphi+\psi}$, *and* $\Phi_\varphi \Phi_\psi = \Phi_{\varphi\psi}$.

Proof. (i): By definition, we get

$$\mathfrak{D}(\alpha\varphi) = \left\{ h \in \mathcal{H} \ \Big| \ \int^* |\alpha\varphi|^2 \, \mathrm{d}\mathfrak{m}_{h,h} < \infty \right\} = \mathfrak{D}(\varphi),$$

and

$$\langle g, \Phi_{\alpha\varphi} h \rangle = \int \alpha\varphi \, \mathrm{d}\mathfrak{m}_{g,h} = \alpha \int \varphi \, \mathrm{d}\mathfrak{m}_{g,h} = \langle g, \alpha \Phi_\varphi h \rangle,$$

for all $g \in \mathcal{H}$ and $h \in \mathfrak{D}(\phi)$.

(ii): $h \in \mathfrak{D}(\Phi_\varphi \Phi_\psi)$ reformulates to $h \in \mathfrak{D}(\psi)$ and $\Phi_\psi h \in \mathfrak{D}(\varphi)$. By Lemma 4.7, $\mathfrak{m}_{g,\Phi_\psi h} = \psi \mathfrak{m}_{g,h}$ for $h \in \mathfrak{D}(\psi)$. We compute

$$\int^* |\varphi| \mathrm{d} |\mathfrak{m}_{g,\Phi_\psi h}| = \int^* |\varphi\psi| \mathrm{d} |\mathfrak{m}_{g,h}|,$$

so

$$\int^{*} |\varphi| \mathrm{d}|\mathfrak{m}_{g,\Phi_\psi h}| < \infty \Leftrightarrow \int^{*} |\varphi\psi| \mathrm{d}|\mathfrak{m}_{g,h}| < \infty.$$

Therefore,

$$g \mapsto \int^{*} \varphi \,\mathrm{d}\mathfrak{m}_{g,\Phi_\psi h} \text{ is continuous } \Leftrightarrow g \mapsto \int^{*} \varphi\psi \,\mathrm{d}\mathfrak{m}_{g,h} \text{ is continuous.}$$

The last statement, reformulates to $h \in \mathfrak{D}(\psi)$ and $\Phi_\psi h \in \mathfrak{D}(\varphi)$ which is equivalent to $h \in \mathfrak{D}(\psi)$ and $h \in \mathfrak{D}(\varphi\psi)$.
As $\langle g, \Phi_\varphi \Phi_\psi h\rangle = \int \varphi \,\mathrm{d}\mathfrak{m}_{g,\Phi_\psi h} = \int \varphi\psi \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_{\varphi\psi} h\rangle$, we get $\Phi_\varphi \Phi_\psi \subset \Phi_{\varphi\psi}$. The proof of $\overline{\Phi_\varphi \Phi_\psi} = \Phi_{\varphi\psi}$ is analogous to the proof of $\overline{\Phi_\varphi + \Phi_\psi} = \Phi_{\varphi+\psi}$ and will be omitted.

(*iii*): To show $\mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi) \subset \mathfrak{D}(\varphi + \psi)$. Let $h \in \mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi)$. By Lemma 4.10,

$$\int^{*} |\varphi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}, \int^{*} |\psi|^2 \,\mathrm{d}\mathfrak{m}_{h,h} < \infty,$$

and by Minkowskys inequality

$$\left(\int^{*} |\varphi + \psi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}\right)^{\frac{1}{2}} \leq \left(\int^{*} |\varphi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}\right)^{\frac{1}{2}} + \left(\int^{*} |\psi|^2 \,\mathrm{d}\mathfrak{m}_{h,h}\right)^{\frac{1}{2}}.$$

Thus $h \in \mathfrak{D}(\varphi + \psi)$. Furthermore, for $g \in \mathcal{H}$

$$\langle g, (\Phi_\varphi + \Phi_\psi) h\rangle = \int \varphi \,\mathrm{d}\mathfrak{m}_{g,h} + \int \psi \,\mathrm{d}\mathfrak{m}_{g,h} = \int (\varphi + \psi) \,\mathrm{d}\mathfrak{m}_{g,h} = \langle g, \Phi_{\varphi+\psi} h\rangle,$$

and thus $\Phi_\varphi + \Phi_\psi \subset \Phi_{\varphi+\psi}$. The rest of the proof will follow after part (*iv*).

(*iv*): $\psi \in \mathrm{L}^\infty(\Phi)$ implies that $\mathfrak{D}(\psi)$ is already the whole space. Thus

$$\mathfrak{D}(\Phi_\varphi \Phi_\psi) = \mathfrak{D}(\varphi\psi) \cap \mathcal{H} = \mathfrak{D}(\varphi\psi),$$

and

$$\mathfrak{D}(\Phi_\varphi + \Phi_\psi) = \mathfrak{D}(\varphi) \cap \mathcal{H} = \mathfrak{D}(\varphi).$$

It follows that

$$\Phi_\varphi \Phi_\psi = \Phi_{\varphi\psi} \text{ and } \Phi_\varphi + \Phi_\psi = \Phi_{\varphi+\psi}.$$

In particular,

$$\mathcal{A} \in \mathscr{E}(\Phi) \Rightarrow E_{\mathcal{A}} \text{ is a projection.}$$

Rest of (*ii*): For $\overline{\Phi_\varphi + \Phi_\psi} = \Phi_{\varphi+\psi}$, we need to show that for $h \in \mathfrak{D}(\varphi + \psi)$, there exists a net $(h_n) \in \mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi)$, such that $\lim h_n = h$, and $\lim(\Phi_\varphi h_n + \Phi_\psi h_n) = \Phi_{\varphi+\psi} h$. Set $\mathcal{A}_n = \{x \in X \mid |\varphi(x)| + |\psi(x)| \leq n\}$. By Lemma 4.10, we have

$$E_{\mathcal{A}_n}(\mathcal{H}) \subset \mathfrak{D}(\varphi) \cap \mathfrak{D}(\psi)$$

and

$$\cup \mathcal{A}_n = X, \ \mathcal{A}_n \subset \mathcal{A}_{n+1}.$$

Thus

$$\lim E_{\mathcal{A}_n} = \mathrm{id}.$$

For all $h \in \mathfrak{D}(\varphi + \psi)$, one has

$$h = \lim E_{\mathcal{A}_n} h =: \lim h_n,$$

and using (iv) combined with the fact that $E_{\mathcal{A}_n}$ is bounded, we get

$$\begin{aligned}\Phi_{\varphi+\psi} h &= \lim E_{\mathcal{A}_n} (\Phi_{\varphi+\psi}) h \\ &= \lim \Phi_{1_{\mathcal{A}_n}(\varphi+\psi)} h \\ &= \lim \Phi_{(\varphi+\psi)1_{\mathcal{A}_n}} h \\ &= \lim \Phi_{\varphi+\psi} E_{\mathcal{A}_n} h \\ &= \lim \Phi_{\varphi+\psi} h_n = \lim(\Phi_\varphi h_n + \Phi_\psi h_n).\end{aligned}$$

□

Lemma 4.14. *For $f \in L^0(\Phi)$, one has*

$$\Phi_f \in \mathcal{B}(\mathcal{H}) \text{ if and only if } f \in L^\infty(\Phi).$$

Proof. "⇐" already proven in Lemma 4.8.

For the other direction we prove that $\|\Phi_f\| \geq \|f\|_\infty$. Let $\lambda < \|f\|_\infty$. Then $\mathcal{A}_\lambda := \{|f| \geq \lambda\}$ is not a Φ-null set. By the polarization identity, there exists a $h \in \mathcal{H}$, such that $\mathcal{A}_\lambda$ is not a $\mathfrak{m}_{h,h}$-null set. Set

$$\mathcal{A}_\lambda = \bigcup_{\substack{\mu \in \mathbb{Q} \\ \mu > \lambda}} \{\lambda \leq |f| \leq \mu\},$$

which gives

$$0 \neq \mathfrak{m}_{h,h}(\mathcal{A}_\lambda) = \sup_{\substack{\mu \in \mathbb{Q} \\ \mu > \lambda}} \mathfrak{m}_{h,h}(\{\lambda \leq |f| \leq \mu\}).$$

Therefore, there exists a $\mu > \lambda$, such that

$$\mathfrak{m}_{h,h}(\{\lambda \leq |f| \leq \mu\}) =: \mathfrak{m}_{h,h}(B) > 0.$$

We then have that $E_B h \in \mathfrak{D}(f)$, since f is bounded on B. Note that this is a priori not true for $\mathcal{A}_\lambda$.

$$\begin{aligned}\|\Phi_f E_B h\|^2 &= \int |f|^2 \, \mathrm{d}\mathfrak{m}_{E_B h, E_B h} \\ &= \int_B |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} \\ &\geq \lambda^2 \mathfrak{m}_{h,h}(B) \\ &= \lambda^2 \int \overline{1}_B 1_B \, \mathrm{d}\mathfrak{m}_{h,h} \\ &= \lambda^2 \int \mathrm{d}\mathfrak{m}_{E_B h, E_B h} = \lambda^2 \|E_B h\|^2\end{aligned}$$

For $\tilde{h} := \frac{E_B h}{\|E_B h\|} \in \mathfrak{D}(f)$, we have $\|\Phi_f \tilde{h}\| \geq \lambda$. Since $\|\tilde{h}\| = 1$, $\|\Phi_f\| \geq \lambda$. □

Lemma 4.15. *For $f \in L^0(\Phi)$, Φ_f is invertible if and only if $\{f = 0\}$ is a Φ-null set and ${}^1\!/_f \in L^\infty(\Phi)$. One then has $\Phi_f^{-1} = \Phi_{1/f}$.*

Proof. $\Leftarrow$: By the previous Lemma $\Phi_{1/f} \in \mathcal{B}(\mathcal{H})$, and by Lemma 4.13

$$\Phi_{1/f}\Phi_f \subset \Phi_{1/f f} = \Phi_1 = \mathrm{id} = \Phi_{f 1/f} = \Phi_f \Phi_{1/f}.$$

Hence, Φ_f is invertible.

$\Rightarrow$: Let $h \in E_{\{f=0\}}(\mathcal{H})$. Since $E_{\{f=0\}}$ is a projection, $E_{\{f=0\}}h = h$. Thus

$$\Phi_f h = \Phi_f E_{\{f=0\}} h = \Phi_{f\cdot 1_{\{f=0\}}} h = \Phi_0 h = 0.$$

Since Φ_f is invertible, $h = 0$ and $E_{\{f=0\}} = 0$ are immediate consequences. Therefore $\{f = 0\}$ is a Φ-null set.

It remains to show that $1/f \in \mathrm{L}^\infty(\Phi)$. We have

$$\Phi_f \cdot \Phi_{1/f} \subset \Phi_1 = I.$$

On $\mathfrak{D}(\Phi_f \cdot \Phi_{1/f})$, it holds that $\Phi_{1/f} = \Phi_f^{-1}$. Since Φ_f is invertible, it is surjective. We claim $\mathcal{H} = \Phi_f(\mathfrak{D}(f)) \subset \mathfrak{D}(\Phi_{1/f})$

$$\begin{aligned}\int \left|\frac{1}{f}\right| \mathrm{d}|\mathfrak{m}_{g,\Phi_f h}| &= \sup_{|\phi|\leq|1/f|} \left|\int \phi \,\mathrm{d}\mathfrak{m}_{g,\Phi_f h}\right| \\ &= \sup_{|\phi|\leq|1/f|} \left|\int \phi f \,\mathrm{d}\mathfrak{m}_{g,h}\right| \leq \int \mathrm{d}|\mathfrak{m}_{g,h}| \leq \|h\|\|g\|.\end{aligned}$$

Thus $\mathfrak{D}(\Phi_{1/f}) = \mathcal{H}$, which implies $1/f \in \mathrm{L}^\infty(\Phi)$, by Lemma 4.14. □

Lemma 4.16. *Let $f \in L^0(\Phi)$. Then*

$$\mathrm{Sp}\,\Phi_f = \bigcap_{E_\mathcal{A}=I} \overline{f(\mathcal{A})}$$

Proof. "$\subset$": Fix $\lambda \in \mathrm{Sp}\,\Phi_f$, $\mathcal{A} \subset X$ such that $E_\mathcal{A} = I$. We claim that $\lambda \in \overline{f(\mathcal{A})}$. By Lemmata 4.13 and 4.15, $\Phi_f - \lambda I = \Phi_{f-\lambda}$ is not invertible implies that either

a) $\{f - \lambda = 0\}$ is not a Φ-null set.

b) $\{f - \lambda = 0\}$ is a Φ-null set, but $1/(f-\lambda) \notin \mathrm{L}^\infty(\Phi)$.

Suppose *a)* holds. Since $\mathcal{A}^c$ is a Φ-null set, $\mathcal{A} \cap \{f = \lambda\}$ is non empty. This means, there exists a $x \in \mathcal{A}$ such that $f(x) = \lambda$, that is $\lambda \in f(\mathcal{A})$. Now suppose that *b)* holds. By Lemma 4.15 $1/(f-\lambda) \notin \mathrm{L}^\infty(\Phi)$ implies that $1/(f-\lambda)$ is unbounded on $\mathcal{A} \setminus \{f = \lambda\}$. Thus, there exists a sequence $(x_n) \in \mathcal{A} \setminus \{f = \lambda\}$, such that

$$\lim |f(x_n) - \lambda| = 0,$$

which implies that

$$\lambda = \lim f(x_n) \in \overline{f(\mathcal{A})}.$$

"$\supset$" Fix $\lambda \notin \mathrm{Sp}(\Phi_f)$. We have to show that there exists a set $\mathcal{A}_0$, $E_{\mathcal{A}_0} = I$ and $\lambda \notin \overline{f(\mathcal{A}_0)}$. By Lemma 4.15, $\Phi_f - \lambda I = \Phi_{f-\lambda}$ is invertible implies that $\{f = \lambda\}$ is a Φ-null set, and $1/f-\lambda \in \mathrm{L}^\infty(\Phi)$. Thus with $M := \|1/f-\lambda\|_\infty$, $\{|f - \lambda| < 1/M\}$ is a Φ-null set. We set $\mathcal{A}_0 := \{|f - \lambda| \geq 1/M\}$. Then $E_{\mathcal{A}_0} = I$ and $d(\lambda, f(\mathcal{A}_0)) \geq 1/M$, that is

$$\lambda \notin \overline{f(\mathcal{A}_0)}.$$

□

Summing up the previous lemmata, we get our

Theorem 4.17 (Main theorem)**.**

1. *If* $f \in L^\infty(\Phi)$*, then* $\mathfrak{D}(f) = \mathcal{H}$*,* $\Phi_f \in \mathscr{L}(\mathcal{H})$ *and*
$$\|\Phi_f\| = \|f\|_\infty = \inf\left\{\alpha > 0 \;\middle|\; \alpha \geq |f|,\ \Phi\text{-}a.e.\right\}.$$
2. *Let* $f \in L^0(\Phi), (f_n)$ *a net in* $L^\infty(\Phi)$*, and* $\alpha, \beta \geq 0$ *such that*
$$|f_n| \leq \alpha|f| + \beta \text{ for all } n$$
$$f_n \to f \ \Phi\text{-}a.e..$$
Then the following statements about $h \in \mathcal{H}$ *are equivalent:*
   (i) $h \in \mathfrak{D}(f)$
   (ii) $\int^* |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} < \infty$
   (iii) $(\Phi_{f_n} h)$ *converges in* $\mathcal{H}$
One then has
$$\|\Phi_f h\|^2 = \int |f|^2 \, \mathrm{d}\mathfrak{m}_{h,h} \text{ and}$$
$$\Phi_f h = \lim \Phi_{f_n} h.$$
3. *For each* $f \in L^0(\Phi)$*,* Φ_f *is a normal Operator, and we have*
$$\Phi_f^* = \Phi_{\bar{f}}.$$
If f is real valued, then Φ_f *is self-adjoint, and* $E_A := \Phi_{1_A}$ *for* $A \in \mathscr{E}(X)$ *is an orthogonal projection.*
4. *(i)* $f \in L^\infty(\Phi)$*,*
$$f \geq 0 \ \Phi\text{-}a.e. \Rightarrow \Phi_f \geq 0$$
   (ii) $\mathcal{A} \in \mathscr{E}(X)$*,*
$$E_\mathcal{A} = 0 \Leftrightarrow \mathcal{A} \ \Phi\text{-null set}$$
   (iii) $U \in \mathscr{E}(X)$ *open,*
$$U \neq \varnothing \Rightarrow E_U \neq 0$$
5. *For each* $f, g \in L^0(\Phi)$*,* $\alpha \in \mathbb{C}$*, we have*
   (i) $\Phi_{\alpha f} = \alpha \Phi_f$
   (ii) $\overline{\Phi_f + \Phi_g} = \Phi_{f+g}$
   (iii) $\mathfrak{D}(\Phi_f \Phi_g) = \mathfrak{D}(fg) \cap \mathfrak{D}(g)$*, and* $\overline{\Phi_f \Phi_g} = \Phi_{fg}$
   (iv) if $g \in L^\infty(\Phi)$ *then* $\Phi_f + \Phi_g = \Phi_{f+g}$*, and* $\Phi_f \Phi_g = \Phi_{fg}$*.*
6. *For* $f \in L^0(\Phi)$*, one has*
$$\Phi_f \in \mathscr{L}(\mathcal{H}) \text{ if, and only if, } f \in L^\infty(\Phi).$$
7. *For* $f \in L^0(\Phi)$*, we have*
$$\operatorname{Sp} \Phi_f = \bigcap_{A \in \mathscr{E}(X)} \overline{f(A)};$$
where, A runs over all $A \in \mathscr{E}(X)$ *such that* $E_A = I$ *and* $A \subset \mathfrak{D}(f)$*.*

5 Spectral Theorem for Unbounded Operators

Using the results from Section 4, we construct a spectral measure for unbounded normal operators, completing the process started in Section 3. We follow [4].

Theorem 5.1 (Spectral Theorem)**.** *Let T be a normal operator on $\mathcal{H}$. Then $\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}} = \operatorname{Sp} T \cup \{\infty\}$ if, and only if T is unbounded. Furthermore $\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}} = \theta(\operatorname{Sp}\mathfrak{A})$, where $\mathfrak{A}(T) := \langle I, A, B, \rangle\,,\ A := (I + T^*T)^{-1},\ B := TA$.*
There exists a uniquely determined spectral measure

$$\Phi\colon L^0\left(\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}}\right) \to \mathscr{L}(\mathcal{H})$$

such that

(i) $\{\infty\}$ is a Φ-zeroset,

(ii) $\Phi_{\mathrm{id}} = T$, where $\mathrm{id}(\infty) := 0$, which is arbitrary.

In this context, spectral measure shall mean, that the map Φ restricted to $L^\infty\left(\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}}\right) \to \mathcal{B}(\mathcal{H})$, is an isomorphism onto its image. For $f \in L^0\left(\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}}\right)$, $\Phi(f)$ has to be a normal, densely defined operator on $\mathcal{H}$. Furthermore, Φ must extend the unique Spectral measure $\Phi\colon \mathrm{C}\left(\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}}\right) \to \mathcal{B}(\mathcal{H})$ from Proposition 3.7.

Proof. In Proposition 3.7, we defined the inverse Gelfand isomorphism

$$\Phi\colon\ \mathrm{C}(\theta(\operatorname{Sp}\mathfrak{A})) \to \mathfrak{A} \subset \mathscr{L}(\mathcal{H}),\ g \mapsto \mathcal{G}^{-1}(g \circ \theta).$$

It follows that Φ is a spectral measure for $\mathrm{C}(\theta(\operatorname{Sp}\mathfrak{A}))$ such that

$$A = \Phi_a,\ a\colon \lambda \mapsto \frac{1}{1+|\lambda|^2},$$
$$B = \Phi_b,\ b\colon \lambda \mapsto \frac{\lambda}{1+|\lambda|^2}.$$

Following [5, Ch. 4.5], we extend Φ to measurable functions. As remarked just before Definition 4.5, this expansion is unique. Furthermore, Theorem 4.17 holds, proving that Φ is a spectral measure.

To show that $\{\infty\}$ is a Φ-nullset, it suffices to show that $E_{\{\infty\}} = 0$. We have

$$AE_{\{\infty\}} = \Phi_a \Phi_{1_{\{\infty\}}} = \Phi_{a1_{\{\infty\}}} = \Phi_0 = 0.$$

Because A is the inverse of $1 + T^*T$, we get

$$E_{\{\infty\}} = (1 + T^*T)AE_{\{\infty\}} = 0.$$

Define $\mathrm{id}\colon \theta(\operatorname{Sp}\mathfrak{A}) \subset \overline{\mathbb{C}} \to \mathbb{C}$, via $\mathrm{id}(\infty) = 0$. Thus,

$$\mathrm{id} \in \mathrm{L}^0(\Phi), \text{ and } (1 + |\,\mathrm{id}\,|^2)a = 1 \quad \Phi\text{-a.e.}$$

Using the fact that a is bounded and Lemma 4.13 (iv), we get

$$I = \Phi_1 = \Phi_{(1+|\,\mathrm{id}\,|^2)a} = \Phi_{(1+|\,\mathrm{id}\,|^2)}\Phi_a = (I + T^*T)A.$$

Furthermore,

$$\begin{aligned} I + TT^* =& \Phi_{(1+|\,\mathrm{id}\,|^2)} \Phi_a (I + TT^*) \\ =& \Phi_{(1+|\,\mathrm{id}\,|^2)} A (I + TT^*) \\ \subset& \Phi_{(1+|\,\mathrm{id}\,|^2)}, \end{aligned}$$

and using again Lemma 4.14, we get

$$T = (I + T^*T)AT \subset \Phi_{(1+|\,\mathrm{id}\,|^2)} TA = \Phi_{(1+|\,\mathrm{id}\,|^2)} B = \Phi_{(1+|\,\mathrm{id}\,|^2)} \Phi_b = \Phi_{\mathrm{id}}.$$

Since T is normal and hence closed, we get that $T = \Phi_{\mathrm{id}}$. It still remains to show that

$$\theta(\mathrm{Sp}\,\mathfrak{A}) = \overline{\mathrm{Sp}\,T}^{\mathbb{C}}.$$

By Lemma 4.16 , we have

$$\mathrm{Sp}\,T = \mathrm{Sp}(\Phi_{\mathrm{id}}) = \bigcap_{E_U = I} \overline{\mathrm{id}(U)}^{\mathbb{C}}.$$

Let $U \subset \theta(\mathrm{Sp}\,\mathfrak{A})$, such that U^{c} is a Φ-nullset. Then

$$\begin{aligned} \mathrm{id}(U) &= U && \text{if } \infty \notin U \\ &= (U \cup \{0\}) \setminus \{\infty\} && \text{if } \infty \in U, \end{aligned}$$

giving us

$$\begin{aligned} \overline{\mathrm{id}(U)}^{\mathbb{C}} &= \overline{U}^{\mathbb{C}} && \text{if } \infty \notin U \\ &= (\overline{U}^{\mathbb{C}} \cup \{0\}) \setminus \{\infty\} && \text{if } \infty \in U. \end{aligned}$$

By Corollary 4.12 , U^{c} does not contain any open sets. Thus

$$\overline{U}^{\mathbb{C}} = \theta(\mathrm{Sp}\,\mathfrak{A}) \setminus \{\infty\},$$

and for each U, such that $E_U = I$

$$\theta(\mathrm{Sp}\,\mathfrak{A}) \setminus \{\infty\} \subset \overline{\mathrm{id}(U)}^{\mathbb{C}} \subset (\theta(\mathrm{Sp}\,\mathfrak{A}) \cup \{0\}) \setminus \{\infty\}.$$

For $U_0 := \theta(\mathrm{Sp}\,\mathfrak{A}) \setminus \{\infty\}$, it holds that $E_{U_0} = I$ and $\mathrm{id}(U_0) = U_0$, and therefore, it follows that

$$\mathrm{Sp}\,T = \theta(\mathrm{Sp}\,\mathfrak{A}) \setminus \{\infty\}.$$

If T is bounded, $\mathrm{Sp}\,T$ is bounded as well, and hence compact, which gives

$$\overline{\mathrm{Sp}\,T}^{\mathbb{C}} = \mathrm{Sp}\,T.$$

Since T is bounded, $I + T^*T = A^{-1}$ is bounded as well, and since $I + T^*T$ is invertible in $\mathcal{B}(\mathcal{H})$ it is invertible in $\mathfrak{A}$ by Proposition 2.1. Thus $I + T^*T$ lies in $\mathfrak{A}$. Assume χ_∞ is an element of $\mathrm{Sp}(\mathfrak{A})$. $I = (I + T^*T)A$ now implies

$$1 = \chi_\infty(I) = \chi_\infty(I + T^*T)\chi_\infty(A) = \chi_\infty(I + T^*T) \cdot 0 = 0,$$

a contradiction. Thus χ_∞ is not an element of $\mathrm{Sp}(\mathfrak{A})$, implying that $\mathrm{Sp}\,T = \theta(\mathrm{Sp}\,\mathfrak{A})$, if T is bounded. If on the other hand, $\mathrm{Sp}\,T$ is compact in $\mathbb{C}$, then

$$\mathrm{id} \in \mathrm{L}^\infty(\Phi), \text{ and so } T = \Phi_{\mathrm{id}} \in \mathcal{B}(\mathcal{H}).$$

Thus, we have proven

$$T \text{ is bounded } \Leftrightarrow \operatorname{Sp} T \text{ compact in } \mathbb{C}.$$

If now T is unbounded, then $\operatorname{Sp} T$ is not compact in $\mathbb{C}$, and hence

$$\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}} = \operatorname{Sp} T \cup \{\infty\}.$$

Since $\theta(\operatorname{Sp} \mathfrak{A}) \subset \overline{\mathbb{C}}$ is compact, we get

$$\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}} = \theta(\operatorname{Sp} \mathfrak{A}).$$

This proves, that there exists a spectral measure $\Phi\colon \mathrm{C}(\overline{\operatorname{Sp} T}^{\overline{\mathbb{C}}}) \to \mathscr{L}(\mathcal{H})$, such that $\Phi_{\mathrm{id}} = T$, and $\{\infty\}$ is a Φ-nullset. To check uniqueness, let Φ' be another spectral measure, such that ∞ is a Φ'-zeroset, and $\Phi'(\mathrm{id}) = T$. To prove $\Phi = \Phi'$, by the theorem of Stone-Weierstra, we only need to show

$$\Phi'_a = \Phi_a = A, \; \Phi'_b = \Phi_b = B.$$

We know

$$T^*T = (\Phi'_{\mathrm{id}})^* \Phi'_{\mathrm{id}} \subset \Phi_{|\,\mathrm{id}\,|^2}.$$

Since T^*T is normal and hence closed, we have equality.

$$\begin{aligned} \Phi'_a(I + T^*T) &= \Phi'_a \Phi'_{1+|\,\mathrm{id}\,|^2} \subset \Phi'_{a(1+|\,\mathrm{id}\,|^2)} = I \\ &= \Phi'_{(1+|\,\mathrm{id}\,|^2)a} = \Phi'_{(1+|\,\mathrm{id}\,|^2)} \Phi'_a = (I - T^*T)\Phi'_a, \end{aligned}$$

that is

$$\begin{aligned} \Phi'_a &= (I + T^*T)^{-1} = A \\ \Phi'_b &= \Phi'_{\mathrm{id}\,\cdot a} = \Phi'_{\mathrm{id}} A = TA = B. \end{aligned}$$

□

6 Applications

Our applications are motivated by quantum physics. Two fundamental operators in quantum mechanics are the momentum operator $P \coloneqq i\frac{\partial}{\partial x}$, and the position operator $M_x(f)(x) = xf(x)$.

Remark 6.1. The study of these operators relies heavily on the chosen Hilbert space $\mathcal{H}$, as $\operatorname{Sp} M_x$ depends on it.

Example 6.2. Let $\mathcal{H} \coloneqq \mathrm{L}^2([0,1])$. $M_x \in \mathcal{B}(\mathcal{H})$ by Hlder's inequality: Let $f \in \mathcal{H}$

$$\|M_x(f)\|^2 = \int_0^1 x^2|f(x)|^2 \,\mathrm{d}\,x \leq 1 \cdot \int_0^1 |f(x)|^2 \,\mathrm{d}\,x = \|f\|^2.$$

Thus $\|M_x\| \leq 1$. Furthermore M_x is self-adjoint, which implies that $\operatorname{Sp} M_x \subset [-1,1]$. But for $\lambda < 0$, $M_{x-\lambda}$ is invertible as $M_{1/(x-\lambda)} \in \mathcal{B}(\mathcal{H})$, again by Hlder. Therefore $\operatorname{Sp} M_x \subset [0,1]$. To show $\operatorname{Sp} M_x \supset [0,1]$, let $\lambda \in [0,1]$. Again the inverse of $M_{x-\lambda}$ would be $M_{1/(x-\lambda)}$, the latter not being bounded, as $\frac{1}{x-\lambda} \notin \mathrm{L}^2([0,1])$. We conclude that x does not a have a preimage in $\mathcal{H}$ under $M_{x+\lambda}$. Hence $\operatorname{Sp} M_x = [0,1]$.

Our Spectral Theorem 2.3 states, that the map $\Phi\colon \mathrm{C}([0,1]) \to \mathrm{Sp}(\langle M_x, I\rangle)$, is an isometry. This isometry sends id to M_x, and the constant function 1 to I. We conclude that ϕ is mapped via Φ to $M_{\phi(x)}\colon g \mapsto \phi \cdot g$. In this example, it is obvious that L^∞ is mapped to $\mathcal{B}(\mathcal{H})$. If we take $\phi(x) = {}^1\!/\!_x$, $\mathfrak{D}(\phi)$ is by definition, all elements $f \in \mathcal{H}$, such that $g \mapsto \langle g, M_\phi f\rangle$ is continuous in g for all $g \in \mathcal{H}$. Since we have such an explicit form of the operator, one readily sees via Lemma 4.10 (*ii*) that $\mathfrak{D}(\phi) = \{f \in \mathcal{H} \mid \|M_\phi(f)\|^2 < \infty\}$. As seen in Example 2.5, or by direct calculation, M_ϕ is the inverse to M_x. M_ϕ is unbounded, and M_x is the bounded inverse to M_x. This is an example, where we started with a bounded operator, and via the Functional Calculus for measurable functions, got an unbounded operator.

If we change the Hilbert space to $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$, the spectrum of our multiplication operator changes to $\mathrm{Sp}(M_x) = \mathbb{R}$. Thus, M_x is unbounded. Its inverse is again $M_{1/x}$. Note that both operators are unbounded, and hence, neither is boundedly invertible.

Example 6.3. Let $\mathcal{H} \coloneqq \mathrm{L}^2([0,1])$. The operator we want to consider is the momentum operator $P = i\frac{\partial}{\partial x}$. We recall the definition of a normal operator. P is called normal if $\mathfrak{D}(P) = \mathfrak{D}(P^*)$ and $\|P\| = \|P^*\|$. To get a normal operator, we need to specify boundary conditions on $\mathfrak{D}(P)$. One easily sees that P^* acts the same way as P. Therefore if P is normal, it is self-adjoint. We compute

$$\begin{aligned}\langle f, Pg\rangle &= \int_0^1 \overline{f(x)} i \frac{\partial}{\partial x}(g(x))\mathrm{d}x = \overline{f(x)} i g(x)\Big|_0^1 - \int_0^1 i\frac{\partial}{\partial x}(\overline{f(x)})g(x)\mathrm{d}x \\ &= i(\overline{f(1)}g(1) - \overline{f(0)}g(0)) + \int_0^1 \frac{\partial}{\partial x}(\overline{if(x)})g(x)\mathrm{d}x \\ &= i(\overline{f(1)}g(1) - \overline{f(0)}g(0)) + \langle Pf, g\rangle\,.\end{aligned}$$

For P to be self-adjoint, the left term from the last line needs to vanish. Thus, we get $\mathfrak{D}(P) = \{f \in H^1([0,1]) \mid f(0) = f(1)\}$. To determine the spectrum of the operator, we look at the eigenvectors of $(P + \lambda I)$. $(P + \lambda I)(f) = f$ reformulates to $f' = (\lambda - 1)f$, giving $f(x) = \mathrm{e}^{i(\lambda-1)x}$. This is in the domain of P, if $\lambda - 1 = 2\pi k$ for $k \in \mathbb{Z}$. We claim that these elements already form a Hilbert basis. To see this, we periodically extend any $f \in \mathcal{H} \coloneqq \mathrm{L}^2([0,1])$ to $\mathrm{L}^2(\mathbb{R})$. Now we can consider functions on the quotient $\mathbb{R} \backslash \mathbb{Z} = \mathbb{S}^1$. For $\mathrm{L}^2(\mathbb{S}^1)$, $\left(\mathrm{e}^{2\pi i k x}\right)_{k \in \mathbb{Z}}$ forms a Hilbert basis, via Fourier expansion. A full proof can be found in [10, Ch. V.4]. Using the Fourier transform $\mathcal{F}$, P becomes the multiplication operator $M_{2\pi k}$: For $f \in \mathfrak{D}(P)$, we get

$$
\begin{aligned}
(\mathcal{F}(Pf))(k) &= \frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{2\pi i k x}\, i(\partial_x f)(x) \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^1 2\pi k\, \mathrm{e}^{2\pi i k x}\, f(x) \mathrm{d}x \\
&= 2\pi k (\mathcal{F} f)(k).
\end{aligned}
$$

The Fourier transform maps $\mathrm{L}^2(\mathbb{S}^1)$ isometrically to $\ell^2(\mathbb{Z})$, found in [8, p. 205]. $\mathrm{Sp}(M_{2\pi k}) = 2\pi\mathbb{Z}$, as $(M_{2\pi k} + \lambda I)^{-1} = (M_{2\pi k + \lambda})^{-1} = M_{(2\pi k + \lambda)^{-1}}$, for $\lambda \notin 2\pi\mathbb{Z}$.

The map $\Phi \colon \mathrm{L}^0(\mathrm{Sp}(P)) \to \mathscr{L}(\mathcal{H}),\ f \mapsto \mathcal{F} M_{f(2\pi k)} \mathcal{F}^{-1}$ satisfies the conditions of being a spectal measure for the operator P. By Theorem 5.1 it is the unique spectral measure for P. We can therefore apply the arguments from Example 6.2, to $M_{2\pi k}$, and transform back.

The fact that we could find another Hilbert space, such that our operator acts as a multiplication operator, was not mere chance. A more general version of the spectral theorem states the following:

Theorem 6.4 (Spectral Theorem, multiplication operator)**.** *Let T be a normal operator on a Hilbert space $\mathcal{H}$. There exists a measure space (Ω, Σ, μ), a measurable function $f \colon \Omega \to \mathbb{C}$, and a unitary operator $U \colon \mathcal{H} \to L^2(\mu)$, such that*

$$
UTU^* \phi = f \cdot \phi = M_f(\phi) \text{ for all } \phi \in \mathfrak{D}(M_f) = \left\{ \phi \in L^2(\mu) \mid f\phi \in L^2(\mu) \right\}.
$$

The proof for bounded self-adjoint T can be found in [10, Ch. VII.1]. Here, we sketch how one can extend this proof for normal operatorrs.

Proof. First assume, that x is a *-cyclic vector for T, meaning that the span of $\left\{ T^{*k} T^n x \mid k, n \in \mathbb{N} \right\}$ lies dense in $\mathcal{H}$. Consider the map $V \colon \mathrm{C}(\mathrm{Sp}(T)) \to \mathcal{H}$, $\phi \mapsto \Phi_\phi x$, where Φ is the unique spectral measure obtained in Theorem 5.1. Then

$$
\int |\phi|^2 \,\mathrm{d}\mathfrak{m}_{x,x} = \int \overline{\phi}\phi \,\mathrm{d}\mathfrak{m}_{x,x} = \left\langle \Phi_{\overline{\phi}\phi} x, x \right\rangle = \langle \Phi_\phi x, \Phi_\phi x \rangle = \|\Phi_\phi x\|^2,
$$

implying that V can be isometrically extended to a map $\overline{V} \colon \mathrm{L}^2(\mathfrak{m}_{x,x}) \to \mathcal{H}$. As x is a cyclic vector, the image of $\overline{V}$ is the whole Hilbert space $\mathcal{H}$. Thus, $\overline{V}$ is a surjective isometry, and therefore unitary. For $\phi \in \mathrm{L}^2(\mathfrak{m}_{x,x})$, such that $\mathrm{id} \cdot \phi$ is still square integrable, we calculate

$$
T(\overline{V}(\phi)) = T(\Phi_\phi x) = (T \circ \Phi_\phi)x = (\Phi_{\mathrm{id}} \Phi_\phi)x = \overline{V}(\mathrm{id} \cdot \phi),
$$

where the last equality holds by Lemma 4.13. This gives

$$
(\overline{V}^{-1} T \overline{V})\phi = \mathrm{id} \cdot \phi.
$$

Now $U \coloneqq \overline{V}^{-1} = \overline{V}^{*}$, and id satisfy the conditions of the theorem.

Unfortunately, we can not expect T to have a cyclic vector. However, we can decompose $\mathcal{H}$ into cyclic subspaces via an argument using Zorn's Lemma. For this argument to work, we need that $\bigcap_k \mathfrak{D}(T^k)$ is dense in $\mathcal{H}$, which is an easy application of Theorem 5.1. We adopt the notation from [10, pp. 337], writing $\mathcal{H} = \bigoplus_2 \mathcal{H}_i$, $x = (x_i)$ for a sum of pairwise orthogonal subspaces, such that the closure of the span is the whole space. Once we have a decomposition $\mathcal{H} = \bigoplus_2 \mathcal{H}_i$ into cyclic subspaces, we apply the considerations above, and get unitary maps $U_i \colon \mathrm{L}^2(\mathfrak{m}_{x_i,x_i}) \to \mathcal{H}_i$, $(U_i T_i U_i^*)\phi_i = f_i \cdot \phi_i$. We now take the direct sum (defined in [2, 214L]) of the measure spaces $(\mathrm{Sp}(T_i), \mathfrak{m}_{x_i,x_i})$, and obtain a new measure space $(\dot{\bigcup} \mathrm{Sp}(T_i), \mathfrak{m})$. We write $f = (f_i)$, $f(x) = f_i(x)$ if $x \in \mathrm{Sp}(T_i)$. Define a new operator $U \colon \mathcal{H} \to \mathrm{L}^2(\mathfrak{m})$, via $U((x_i)) = (U_i(x_i))$. It follows that U is unitary, and $(UTU^*)(\phi) = f\phi$, finishing the proof. □

The theorem tells us, that the multiplication operator is indeed the most general, and in a certain sense, the only example of a normal operator.

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