Representation Theory I

A short collection of basics fun facts about Lie algebras and their representations

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Preface

The following text is based on my notes for the lecture *Representation Theory I* which was given by Prof. Dr. Stroppel during the summer term 2015 at the University of Bonn. The lecture is mostly about the finite dimensional representation theory of finite dimensional semisimple complexe Lie-Algebras. No previous knowledge about Lie-Algebras is required. At a few points I also took some motivation from [**Humphreys**], mostly to clarify some proofs.

These notes have mainly two purposes: One is to prepare myself for the exam, as typing out these notes forces myself to go trough them at a slow speed, paying much attention to details and look things up if necessary. The other is to allow the other students to learn with some nicely typed out text instead of a bunch of handwritten (and sometimes non-existing) notes.

Contents

1. The Basics

1.1. Basic definitions

1.1.1. Definition and examples of Lie algebras

Definition 1.1.1. Let $\mathfrak{g}$ be a vector space over some field k. A k-bilinear map

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is called a *Lie bracket* if it satisfies the following two conditions:

1. $[\cdot,\cdot]$ is *alternating*, i.e. $[x,x] = 0$ for every $x \in \mathfrak{g}$.
2. $[\cdot,\cdot]$ satisfies the *Jacobi identity*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{for all } x,y,z \in \mathfrak{g}.$$

A k-vector space $\mathfrak{g}$ together with the Lie-bracket $[\cdot,\cdot]$ is called a *k-Lie algebra*.

Remark 1.1.2. Any Lie bracket $[\cdot,\cdot]$ is antisymmetric, i.e. $[y,x] = -[x,y]$ for all $x,y \in \mathfrak{g}$, because

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x].$$

Remark 1.1.3. Using the antisymmetry of the Lie bracket the Jacobi identity can be rewritten as

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \quad \text{for all } x,y,z \in \mathfrak{g}.$$

Examples 1.1.4. 1. Any vector space V becomes a Lie algebra via

$$[x,y] = 0 \quad \text{for all } x,y \in V.$$

2. Any *associative* k-algebra A becomes a Lie algebra via

$$[a,b] = ab - ba \quad \text{for all } a,b \in A.$$

Then $[\cdot,\cdot]$ is alternating and because A is associative it follows that for all $a,b,c \in A$

$$\begin{aligned}
&[a,[b,c]] + [b,[c,a]] + [c,[a,b]] \\
&= [a,(bc-cb)] + [b,(ca-ac)] + [c,(ab-ba)] \\
&= a(bc-cb) - (bc-cb)a + b(ca-ac) - (ca-ac)b + c(ab-ba) - (ab-ba)c \\
&= abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\
&= 0.
\end{aligned}$$

The following two are important examples of this construction:

a) The k-algebra of $(n \times n)$-matrices $\mathrm{M}_n(k)$ becomes a Lie algebra via

$$[A, B] = AB - BA \quad \text{for all } A, B \in \mathrm{M}_n(k).$$

This is called the *general linear Lie algebra* and is denoted by $\mathfrak{gl}_n(k)$.

b) More generally for any k-vector space the k-algebra $\mathrm{End}_k(V)$ becomes a Lie algebra via

$$[\varphi_1, \varphi_2] \coloneqq \varphi_1 \circ \varphi_2 - \varphi_2 \circ \varphi_1 \quad \text{for all } \varphi_1, \varphi_2 \in \mathrm{End}_k.$$

This is called the *general linear Lie algebra for V* and is denoted by $\mathfrak{gl}(V)$.

Definition 1.1.5. Let $\mathfrak{g}$ be a k-Lie algebra. A *Lie subalgebra* of a $\mathfrak{g}$ is a k-linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that

$$[x, y] \in \mathfrak{h} \quad \text{for all } x, y \in \mathfrak{h}.$$

An *ideal* inside $\mathfrak{g}$ is a k-linear subspace $I \subseteq \mathfrak{g}$ such that

$$[x, y] \in I \quad \text{for all } x \in \mathfrak{g} \text{ and } y \in I.$$

That I is an ideal in $\mathfrak{g}$ is denoted by $I \trianglelefteq \mathfrak{g}$.

Remark 1.1.6. It is not necessary to distinguish between left ideals or right ideals in a Lie algebra because the Lie bracket is antisymmetric.

Remark 1.1.7. For a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ it follows that $\mathfrak{h}$ becomes a Lie algebra by restricting the Lie bracket of $\mathfrak{g}$ to $\mathfrak{h}$. Every ideal inside $\mathfrak{g}$ is also a subalgebra of $\mathfrak{g}$.

Definition 1.1.8. Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is

$$Z(\mathfrak{g}) \coloneqq \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for every } y \in \mathfrak{g}\}$$

$\mathfrak{g}$ is called *abelian* if $Z(\mathfrak{g}) = 0$, i.e. if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Definition 1.1.9. For a Lie-algebra $\mathfrak{g}$ over some field k and subsets $X, Y \subseteq \mathfrak{g}$ let

$$[X, Y] \coloneqq \mathrm{span}_k\{[x, y] \mid x \in X, y \in Y\}.$$

Remark 1.1.10. Notice that $\mathfrak{g}$ is abelian if and only if $[\mathfrak{g}, \mathfrak{g}] = 0$. Also notice that $[\mathfrak{g}, \mathfrak{g}]$ and $Z(\mathfrak{g})$ are ideals inside $\mathfrak{g}$.

Lemma 1.1.11. *Let $\mathfrak{g}$ be a Lie algebra over some field k.*

1. *If I_λ, $\lambda \in \Lambda$ is a collection of ideals $I_\lambda \trianglelefteq \mathfrak{g}$ then also $\bigcap_{\lambda \in \Lambda} I_\lambda \trianglelefteq \mathfrak{g}$ and $\sum_{\lambda \in \Lambda} I_\lambda \trianglelefteq \mathfrak{g}$.*
2. *If $I, J \trianglelefteq \mathfrak{g}$ then also $[I, J] \trianglelefteq \mathfrak{g}$.*

Proof. 1. This follows from direct calculation.

2. As $[I, J]$ is spanned by the elements $[y, z]$ with $y \in I$ and $z \in J$ it is enough to show that $[x, [y, z]] \in [I, J]$ for every $x \in \mathfrak{g}$, $y \in I$ and $z \in J$. This follows from $I, J \trianglelefteq \mathfrak{g}$ and the Jacobi identity, because

$$[x, [y, z]] = [\underbrace{[x, y]}_{\in I}, z] + [y, \underbrace{[x, z]}_{\in J}] \in [I, J]. \qquad \square$$

Definition 1.1.12. A Lie algebra $\mathfrak{g}$ is called *linear* if $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V.

Example 1.1.13. 1. Let $\mathfrak{g} = \mathfrak{gl}_n(k)$. Then

$$\mathfrak{sl}_n(k) = \{A \in \mathfrak{g} \mid \operatorname{tr} A = 0\}$$

is an ideal in $\mathfrak{g}$ because

$$\mathfrak{sl}_n(k) = [\mathfrak{g}, \mathfrak{g}].$$

To see this first notice that on the one hand

$$\operatorname{tr}[A, B] = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = \operatorname{tr}(AB) - \operatorname{tr}(AB) = 0$$

for all $A, B \in \mathfrak{g}$ and therefore $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{sl}_n(k)$.

On the other hand notice that $\mathfrak{sl}_n(k)$ has a basis given by the elementary matrices e_{ij} with $1 \leq i \neq j \leq n$ and $e_{11} - e_{ii}$ with $i = 2, \ldots, n$. Each of these matrices is given as a commutator, namely $e_{ij} = [e_{ii}, e_{ij}]$ for $1 \leq i \neq j \leq n$ and $e_{11} - e_{ii} = [e_{1i}, e_{i1}]$ for $i = 2, \ldots, n$. Therefore $\mathfrak{sl}_n(k) \subseteq [\mathfrak{g}, \mathfrak{g}]$.

2. The upper triangular matrices

$$\mathfrak{t}_n(k) := \left\{ \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \middle| \, a_{ij} \in k \text{ for every } 1 \leq i \leq j \leq n \right\}$$

are a Lie subalgebra of $\mathfrak{gl}_n(k)$.

3. The strictly upper triangular matrices

$$\mathfrak{n}_n(k) := \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \middle| \, a_{ij} \in k \text{ for every } 1 \leq i < j \leq n \right\}$$

are a Lie subalgebra of $\mathfrak{t}_n(k)$ and therefore also of $\mathfrak{gl}_n(k)$. It is even an ideal in $\mathfrak{t}_n(k)$ because $\mathfrak{n}_n(k) = [\mathfrak{t}_n(k), \mathfrak{t}_n(k)]$.

Definition 1.1.14. If $\mathfrak{g}$ is a Lie algebra and $U \subseteq \mathfrak{g}$ a linear subspace then

$$N_{\mathfrak{g}}(U) := \{x \in \mathfrak{g} \mid [x,y] \in \mathfrak{g} \text{ for every } y \in U\}$$

is called the *normalizer* of U in $\mathfrak{g}$ and

$$Z_{\mathfrak{g}}(U) := \{x \in \mathfrak{g} \mid [x,y] = 0 \text{ for every } y \in U\}$$

is called the *centralizer* of U in $\mathfrak{g}$. For a single element $x \in \mathfrak{g}$ the centralizer of x in $\mathfrak{g}$ defined as

$$Z_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [x,y] = 0\}.$$

Lemma 1.1.15. *Let $\mathfrak{g}$ be a Lie algebra and $U \subseteq \mathfrak{g}$ a linear subspace. Then $N_{\mathfrak{g}}(U)$ and $Z_{\mathfrak{g}}(U)$ are Lie subalgebras of $\mathfrak{g}$. $Z_{\mathfrak{g}}(x)$ is a Lie subalgebra of $\mathfrak{g}$ for every $x \in \mathfrak{g}$.*

Proof. If $x, y \in N_{\mathfrak{g}}(U)$ then by the Jacobi identity it follows for every $z \in U$ that

$$[[x,y],z] = -[z,[x,y]] = -[[z,x],y] - [x,[z,y]] =\in \mathfrak{h}$$

and therefore $[x,y] \in N_{\mathfrak{g}}(U)$. In the same way it follows for all $x, y \in \mathfrak{g}$ and $z \in U$ that

$$[[x,y],z] = -[[z,x],y] - [x,[z,y]] = 0$$

and therefore $[x,y] \in Z_{\mathfrak{g}}(U)$. For every $x \in \mathfrak{g}$ the span $kx \subseteq \mathfrak{g}$ is a linear subspace with $Z_{\mathfrak{g}}(x) = Z_{\mathfrak{g}}(kx)$, which is why $Z_{\mathfrak{g}}(x)$ is a Lie subalgebra of $\mathfrak{g}$. □

Remark 1.1.16. Let $\mathfrak{g}$ be a Lie algebra and $L \subseteq \mathfrak{g}$ a linear subspace. Then L is a Lie subalgebra if and only if $L \subseteq N_{\mathfrak{g}}(L)$. Then L is not only contained in $N_{\mathfrak{g}}(L)$ but $N_{\mathfrak{g}}(L)$ is the maximal subalgebra of $\mathfrak{g}$ which contains L as an ideal. In particular L is an ideal in $\mathfrak{g}$ if and only if $N_{\mathfrak{g}}(L) = \mathfrak{g}$.

1.1.2. Homomorphisms of Lie algebras

Definition 1.1.17. Given Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ over the same field k a *homomorphism of Lie algebras* $\mathfrak{g}_1 \to \mathfrak{g}_2$ is a k-linear map $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$f([x,y]) = [f(x), f(y)] \quad \text{for all } x, y \in \mathfrak{g}_1.$$

Examples 1.1.18. 1. For any Lie algebra $\mathfrak{g}$ the identity $\mathrm{id}_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.

2. Given Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ and Lie algebra homomorphisms $f_1\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $f_2\colon \mathfrak{g}_2 \to \mathfrak{g}_3$ the composition $f_2 \circ f_1\colon \mathfrak{g}_1 \to \mathfrak{g}_3$ is also a homomorphism of Lie algebras.

3. If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra then the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is a homomorphism of Lie algebras.

4. Given two abelian Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ any linear map $\mathfrak{g}_1 \to \mathfrak{g}_2$ is already a homomorphism of Lie algebras.

5. Let $\mathfrak{g}$ ba a Lie algebra over an arbitrary fielid k. Then for every $x \in \mathfrak{g}$ let

$$\operatorname{ad}(x)\colon \mathfrak{g} \to \mathfrak{g} \quad \text{mit} \quad \operatorname{ad}(x)(y) = [x,y] \quad \text{for every } y \in \mathfrak{g}.$$

Then the map $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras. This follows from the Jacobi identity because for all $x, y, z \in \mathfrak{g}$

$$\begin{aligned}\operatorname{ad}([x,y])(z) &= [[x,y],z] = -[z,[x,y]] = -[[z,x],y] - [x,[z,y]] = [x,[y,z]] - [y,[x,z]] \\ &= \operatorname{ad}(x)\operatorname{ad}(y)(z) - \operatorname{ad}(y)\operatorname{ad}(x)(z) = [\operatorname{ad}(x), \operatorname{ad}(y)](z).\end{aligned}$$

6. If A_1 and A_2 are associative k-algebras and $f\colon A_1 \to A_2$ a homomorphism of k-algebras then it is also a homomorphism of Lie algebras because

$$f([a,b]) = f(ab - ba) = f(a)f(b) - f(b)f(a) = [f(a), f(b)] \quad \text{for all } a, b \in A_1.$$

7. Let $\mathfrak{g}$ be a Lie algebra over an arbitary field k. If $\phi\colon \mathfrak{sl}_2(k) \to \mathfrak{g}$ is a homomorphism of Lie algebras then the images

$$E \coloneqq \phi(e), \quad H \coloneqq \phi(h), \quad F \coloneqq \phi(f)$$

satisfy the relations

$$[H,E] = 2E, \quad [H,F] = 2F, \quad [E,F] = H.$$

On the other hand every triple (E', H', F') of elements satisfying the relations above (with X replaced by X' for $X \in \{E, H, F\}$) gives rise to a unique homomorphism of Lie algebras $\phi'\colon \mathfrak{sl}_2(k) \to \mathfrak{g}$ with

$$\phi'(E) = E', \quad \phi'(H) = H', \quad \phi'(F) = F'.$$

Hence there is a bijection between Lie algebra homomorphisms $\mathfrak{sl}_2(k) \to \mathfrak{g}$ and triples as above. Such triples will play an important part later on.

Definition 1.1.19. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras over the same field k. A homomorphism of Lie algebras $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is called an *isomorphism of k-Lie algebras* if f is bijective.

Lemma 1.1.20. *If $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism of k-Lie algebras, then the k-linear map $f^{-1}\colon \mathfrak{g}_2 \to \mathfrak{g}_1$ is also a homomorphism of Lie-algebras and therefore also an isomorphism.*

Proof. For all $x, y \in \mathfrak{g}_2$

$$\begin{aligned}f^{-1}([x,y]) &= f^{-1}([f(f^{-1}(x)), f(f^{-1}(y))]) \\ &= f^{-1}(f([f^{-1}(x), f^{-1}(y)])) \\ &= [f^{-1}(x), f^{-1}(y)].\end{aligned}$$

□

Remark 1.1.21. It follows that Lie algebras over the same field k together with homomorphisms of Lie algebras and their usual composition form a category, which will be denoted by k-**Lie**. An *isomorphism of k-Lie algebras* is the same as an isomorphism in k-**Lie**.

Example 1.1.22 (Classification of one- and two-dimensional Lie algebras)**.** Let k be any field.

As every linear map between abelian Lie algebras is already a homomorphism of Lie algebras it follows that there exists up to isomorphism exactly one n-dimensional abelian Lie algebra over k for every $n \in \mathbb{N}$.

If $\mathfrak{g}$ is a one-dimensional Lie algebra over k then the Lie bracket of $\mathfrak{g}$ is zero because it is alternating, which is why $\mathfrak{g}$ is abelian. Hence there is up to isomorphism exactly one one-dimensional Lie algebra over k.

Up to isomorphism there exists exactly one two-dimensional abelian k-Lie algebra

Suppose that $\mathfrak{g}$ is a two-dimensional, non-abelian Lie algebra over k. Let $\tilde{x}$,$\tilde{y}$ be a basis of $\mathfrak{g}$. Because $[\mathfrak{g}, \mathfrak{g}]$ is non-abelian it follows that $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and on the other hand $[\mathfrak{g}, \mathfrak{g}]$ is spanned by $[\tilde{x}, \tilde{y}]$ because the Lie bracket is alternating, so $[\mathfrak{g}, \mathfrak{g}] = k[\tilde{x}, \tilde{y}]$ with $[\tilde{x}, \tilde{y}] \neq 0$. Let $x \coloneqq [\tilde{x}, \tilde{y}]$ and $y \in \mathfrak{g}$ such that x, y is a basis of $\mathfrak{g}$. Then $[x, y] \neq 0$ and $[x, y] \in kx$. By rescaling y it can be assumed that $[x, y] = x$. This that up to isomorphism there is at most one two-dimensional, non-abelian Lie algebra $\mathfrak{g}$ over k.

Such an Lie algebra also exists. It can be realized as a subalgebra of $\mathfrak{gl}_2(k)$ by choosing

$$x \coloneqq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_{12} \quad \text{and} \quad y \coloneqq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = e_{22}$$

because

$$[x, y] = [e_{12}, e_{22}] = e_{12}e_{22} - e_{22}e_{12} = e_{12} = x.$$

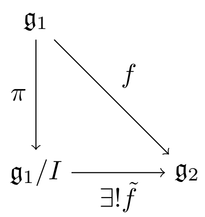
Hence there are up to isomorphism exactly two two-dimensional Lie algebras over k.

Proposition 1.1.23 (Homomorphism theorem)**.** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ a homomorphism of Lie algebras.*

1. $\ker f \trianglelefteq \mathfrak{g}_1$ *is an ideal.*

2. $\operatorname{im} f \subseteq \mathfrak{g}_2$ *is a Lie subalgebra.*

3. *If $I \trianglelefteq \mathfrak{g}_1$ is an ideal with $\ker f \subseteq I$ then there exists a unique homomorphism of Lie algebras $\tilde{f}\colon \mathfrak{g}_1/I \to \mathfrak{g}_2$ with $f = \tilde{f} \circ \pi$ where $\pi\colon \mathfrak{g}_1 \to \mathfrak{g}_1/I$ is the canonical projection.*



4. *If $I, J \trianglelefteq \mathfrak{g}$ are subideals with $I \subseteq J$ then $J/I \trianglelefteq \mathfrak{g}/I$ and the natural isomorphism of vector spaces*

$$(\mathfrak{g}/I)/(J/I) \to \mathfrak{g}/I, \quad (x+I)+(J/I) \mapsto x+I$$

is already a natural isomorphism of Lie algebras.

5. *If $I, J \trianglelefteq \mathfrak{g}$ are subideals then the natural isomorphism of vector spaces*

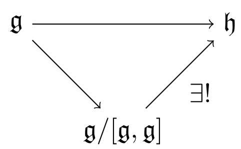
$$(I+J)/J \to I/(I \cap J)$$

defined by

$$(x+J)+I \mapsto x+(I \cap J) \quad \text{for every } x \in I$$

is already a natural isomorphism of Lie algebras.

Remark 1.1.24. For a Lie algebra $\mathfrak{g}$ the ideal $[\mathfrak{g}, \mathfrak{g}]$ is the minimal ideal inside $\mathfrak{g}$ such that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian. Furthermore given any abelian Lie algebra $\mathfrak{h}$ any homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{h}$ factorizes through a unique homomorphism of Lie algebras $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathfrak{h}$.



1.1.3. New Lie algebras from old ones

Definition 1.1.25. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the same field k. Then the *product* of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is defined as the k-vector space $\mathfrak{g}_1 \times \mathfrak{g}_2$ together with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]) \quad \text{for all } (x_1, y_1), (x_2, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

Definition 1.1.26. Let $\mathfrak{g}$ be a Lie algebra and $I \trianglelefteq \mathfrak{g}$. Then the induced Lie algebra structure on the quotient vector space $\mathfrak{g}/I$ is given by

$$[x+I, y+I] = [x, y] + I \quad \text{for all } x, y \in \mathfrak{g}.$$

Remark 1.1.27. The Lie algebra structure on the quotient $\mathfrak{g}/I$ is well-defined: If $x, y, x', y' \in \mathfrak{g}$ with $x + I = x' + I$ and $y + I = y' + I$ then $x - x' \in I$ and $y - y' \in I$ and thus

$$\begin{aligned}[x, y] + I &= [x' + x - x', y' + y - y'] + I \\ &= [x', y'] + \underbrace{[x', y - y']}_{\in I} + \underbrace{[x - x', y']}_{\in I} + \underbrace{[x - x', y - y']}_{\in I} + I = [x', y'] + I.\end{aligned}$$

The additional properties of a Lie bracket follows from the Lie bracket of $\mathfrak{g}$ safisfying them.

Lemma 1.1.28. *1. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras then the canonical projections*

$$\pi_i : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}_i, \quad (x_1, x_2) \mapsto x_i \quad \textit{for } i = 1, 2$$

are homomorphisms of Lie-algebras.

2. If $\mathfrak{g}$ is a Lie algebra and $I \trianglelefteq \mathfrak{g}$ then the canonical projection $\pi : \mathfrak{g} \to \mathfrak{g}/I, x \mapsto [x]$ is a homomorphism of Lie algebras.

Lemma 1.1.29. *Let $\mathfrak{g}$ be a Lie algebra over k and A an associative, commutative k-algebra. Then $A \otimes_k \mathfrak{g}$ is a Lie algebra over k via*

$$[a \otimes x, b \otimes y] = (ab) \otimes [x, y] \quad \textit{for all } a, b \in A \textit{ and } x, y \in \mathfrak{g}.$$

Similarly $\mathfrak{g} \otimes_k A$ carries the structure of a Lie algebra over k via

$$[x \otimes a, y \otimes b] = [x, y] \otimes (ab) \quad \textit{for all } x, y \in \mathfrak{g} \textit{ and } a, b \in A.$$

Example 1.1.30. If L/k is a field extension and $\mathfrak{g}$ a Lie algebra over k, then $L \otimes_k \mathfrak{g}$ is a Lie algebra over k via

$$[\lambda \otimes x, \mu \otimes y] = (\lambda\mu) \otimes [x, y] \quad \text{for alle } \lambda, \mu \in k \text{ and } x, y \in \mathfrak{g}.$$

$L \otimes_k \mathfrak{g}$ also carries the structure of an L-vector space via extension of scalars, i.e.

$$\lambda \cdot (\mu \otimes x) = (\lambda\mu) \otimes x \quad \text{for alle } \lambda, \mu \in k \text{ and } x \in \mathfrak{g},$$

and the Lie bracket is not only k-bilinear, but also L-bilinear. Hence the structure of a k-Lie algebra on $L \otimes_k \mathfrak{g}$ can be extended to the structure of a Lie algebra over L. (Notice that the Jacobi-Identity is independent of the ground field.)

Definition 1.1.31. Let $\mathfrak{g}$ be a Lie algebra and $A = k[t, t^{-1}]$ be the algebra of Laurent polynomials over k. Then

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_k A$$

with the Lie bracket as in Lemma 1.1.29 is called the *loop (Lie) algebra* of $\mathfrak{g}$.

Another example for constructing new Lie algebras out of old ones are *central extensions*: Let $\mathfrak{g}$ be any k-Lie algebra.

$$\tilde{\mathfrak{g}} := \mathfrak{g} \oplus k = \{x + \lambda c \mid x \in \mathfrak{g}, \lambda \in k\},$$

where we understand c as a formal variable. Suppose that $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$ is a k-bilinear map satisfying the following properties:

1. κ is antisymmetric, i.e. $\kappa(x, y) = -\kappa(y, x)$ for all $x, y \in \mathfrak{g}$.

2. κ satisfies the 2-cocycle condition

$$\kappa([x, y], z) + \kappa([y, z], x) + \kappa([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then $\tilde{\mathfrak{g}}$ becomes a Lie algebra via

$$[x + \lambda c, y + \mu c] \coloneqq [x, y] + \kappa(x, y)\lambda\mu c \quad \text{for all } x, y \in \mathfrak{g} \text{ and } \lambda, \mu \in k.$$

Note that c is central in $\tilde{\mathfrak{g}}$ in the sense that $[x, c] = 0$ for all $x \in \mathfrak{g}$.

Example 1.1.32. Let $\mathfrak{g} = \mathfrak{gl}_n(k)$. We define a symmetric bilinear form on $\mathfrak{g}$ via

$$(A, B)_{\mathrm{tr}} = \mathrm{tr}(AB) \quad \text{for all } A, B \in \mathfrak{g}.$$

We define a bilinear form

$$\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to k[t, t^{-1}], \quad (x \otimes p, y \otimes q) \mapsto (x, y)_{\mathrm{tr}}\; pq$$

We now get a 2-cocycle $\kappa\colon \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to k$ via

$$\kappa(a, b) \coloneqq \mathrm{Res}\left(\frac{\partial a}{\partial t}, b\right).$$

κ is also antisymmetric: Let $a = x \otimes t^i$ and $b = y \otimes t^j$ with $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$. Then

$$\begin{aligned}\kappa(x \otimes t^i, y \otimes t^j) = \mathrm{Res}(ix \otimes t^{i-1}, y \otimes t^j) &= \mathrm{Res}(it^{i+j-1}(x, y)_{\mathrm{tr}}) \\ &= \begin{cases} i(x, y)_{\mathrm{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

In the same way we find that

$$\kappa(y \otimes t^j, x \otimes t^i) = \begin{cases} j(x, y)_{\mathrm{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(\cdot, \cdot)_{\mathrm{tr}}$ is symmetric we find that

$$\begin{aligned}\kappa(x \otimes t^i, y \otimes t^j) &= \begin{cases} i(x, y)_{\mathrm{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -j(x, y)_{\mathrm{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= -\kappa(y \otimes t^j, x \otimes t^i).\end{aligned}$$

1.1.4. Derivations

Definition 1.1.33. Let A be a k-algebra (not necessarily unitary of even associative). A *derivation of* A is a k-linear map $d\colon A \to A$ such that

$$d(ab) = d(a)b + ad(b) \quad \text{for all } a, b \in A.$$

We set

$$\mathrm{Der}(A) \coloneqq \{d\colon A \to A \mid d \text{ is a derivation of } A\}.$$

Remark 1.1.34. $\mathrm{Der}(A)$ is a k-linear subspace of $\mathrm{End}_k(A)$.

Example 1.1.35. Let A be a k-algebra. It follows from direct calculation that for all $d, d' \in \mathrm{Der}(A)$ the commutator $[d, d'] = d \circ d' - d' \circ d$ is again a derivation $\mathrm{Der}(A)$. Hence $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Lemma 1.1.36. *Let $\mathfrak{g}$ be a Lie algebra. Then for any $x \in \mathfrak{g}$ the map*

$$\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x, y]$$

is a derivation of $\mathfrak{g}$.

Proof. By the Jacobi identity

$$\begin{aligned}\mathrm{ad}(x)([y,z]) &= [x,[y,z]] = [[x,y],z] + [y,[x,z]] \\ &= [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]\end{aligned}$$

for all $y, z \in \mathfrak{g}$. □

Definition 1.1.37. Let $\mathfrak{g}$ be a Lie algebra. A derivation of $\mathfrak{g}$ is called *inner* if it is of the form $\mathrm{ad}(x)$ for some $x \in \mathfrak{g}$.

Lemma 1.1.38. *If $\mathfrak{g}$ is a Lie algebra then the inner derivations form an ideal inside of* $\mathrm{Der}(\mathfrak{g})$.

Proof. Let $I \coloneqq \operatorname{im}\mathrm{ad} \subseteq \mathrm{Der}(\mathfrak{g})$ be the linear subspace of inner derivations. For any $\delta \in \mathrm{Der}$ and $x \in \mathfrak{g}$ it follows that for any $y \in \mathfrak{g}$

$$\begin{aligned}&[\delta, \mathrm{ad}(x)](y) = (\delta\,\mathrm{ad}(x) - \mathrm{ad}(x)\delta(x))(y) \\ &= \delta([x,y]) - [x, \delta(y)] = [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)] \\ &= [\delta(x), y] = \mathrm{ad}(\delta(x))(y).\end{aligned}$$

Hence $[\delta, \mathrm{ad}(x)] = \mathrm{ad}(\delta(x)) \in I$. □

1.1.5. Simple Lie algebras

Definition 1.1.39. A Lie algebra $\mathfrak{g}$ is *simple* if 0 and $\mathfrak{g}$ are the only ideals inside $\mathfrak{g}$ and $\mathfrak{g}$ is not abelian.

Lemma 1.1.40. *Let $\mathfrak{g}$ be a simple Lie algebra. Then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$.*

Proof. Because $\mathfrak{g}$ is simple it is not abelian. Therefore $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and $Z(\mathfrak{g}) \neq \mathfrak{g}$. Since $[\mathfrak{g}, \mathfrak{g}]$ and $Z(\mathfrak{g})$ are ideals inside $\mathfrak{g}$ it follows that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$. By the homomorphism theorem $\mathfrak{g}$ is isomorphic to its image $\mathrm{ad}\,\mathfrak{g}$ and hence to a linear Lie algebra. □

Corollary 1.1.41. *Let $\mathfrak{g}$ be simple. Then the homomorphism* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto \mathrm{ad}(x)$ *is injective. In particular $\mathfrak{g}$ can be realized as a linear Lie algebra.*

Proof. As part of Examples 1.1.18 has already been shown that ad is a homomorphism of Lie algebras. That it is injective follows directly from $\ker \operatorname{ad} = Z(\mathfrak{g}) = 0$. ☐

It can be shown that every finite dimensional Lie algebra can be realized as a linear Lie algebra. This will not be proven in this lecture and is by far not trivial.

Theorem 1.1.42 (Ado)**.** *Every finite dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a linear Lie algebra.*

Examples 1.1.43. 1. Since $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] = \mathfrak{sl}_n(k) \neq \mathfrak{gl}_n(k)$ we find that $\mathfrak{gl}_n(k)$ is not simple.

2. Let $\mathfrak{g} = \mathfrak{sl}_2(k)$. Then $\mathfrak{g}$ is simple if and only if $\operatorname{char} k \neq 2$. To see this consider the basis (e, h, f) of $\mathfrak{sl}_2(k)$ consisting of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

of $\mathfrak{sl}_2(k)$. Then

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

If $\operatorname{char} k = 2$ then h spans a 1-dimensional ideal, thus $\mathfrak{sl}_2(k)$ is not simple. Suppose that $\operatorname{char} k \neq 2$ and let $I \subseteq \mathfrak{sl}_2(k)$ be an ideal with $I \neq 0$. From the above relations it follows that if I contains one of the basis vectors e, h or f then already $I = \mathfrak{sl}_2(k)$. Let $x \in I$ with $x \neq 0$ and write $x = \alpha e + \beta h + \gamma f$. Then

$$[e, x] = -2\beta e + \gamma h \quad \text{and} \quad [e, [e, x]] = -2\gamma e.$$

Since $\gamma = 0$ or $\gamma \neq 0$ we find that $e \in I$.

Definition 1.1.44. Let k be any field. The basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

of $\mathfrak{sl}_2(k)$ is called the *standard basis* of $\mathfrak{sl}_2(k)$.

Remark 1.1.45. If $\operatorname{char} k = 0$ then $\mathfrak{sl}_n(k)$ is simple for all $n \geq 2$.

1.2. Representations of Lie algebras

1.2.1. Definition and examples

Definition 1.2.1. Let $\mathfrak{g}$ be a k-Lie algebra. A *representation* of $\mathfrak{g}$ is a k-vector space V together with a homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. This representation is called *faithful* if ρ is injective. The *dimension* of this representation is the dimension of V.

Remark 1.2.2. Equivalently a representation of $\mathfrak{g}$ is a k-vector space V together with a k-bilinear map $\mathfrak{g} \times V \to V, (x, v) \mapsto x.v$ such that

$$x.(y.v) - y.(x.v) = [x, y].v \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V. \tag{1}$$

Such an action results in a homomorphism of Lie algebras $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ by setting

$$\rho(x) \colon V \to V, \quad v \mapsto x.v \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

On the other hand any homomorphism $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ an action as above by setting

$$x.v \coloneqq \rho(x)(v) \quad \text{for evevery } x \in \mathfrak{g} \text{ and } v \in V.$$

Both constructions are inverse to each other.

We will not distinguish between these two concepts and choose whichever is more useful in the given situation.

Remark 1.2.3. If $(x_i)_{i \in I}$ is a basis of a Lie algebra $\mathfrak{g}$ then for $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ to be a homomorphism of Lie algebras it is enough to check that $\rho([x_i, x_j]) = [\rho(x_i), \rho(x_j)]$ for all $i, j \in I$. Therefore it also sufficies to check (1) for basis elements, i.e. that

$$x_i.(x_j.v) - x_j.(x_i.v) = [x_i, x_j].v \quad \text{for all } i, j \in I \text{ and } v \in V.$$

Remark 1.2.4. Ado's theorem is equivalent to every finite dimenisonal Lie algebra having a faithful representation.

Examples 1.2.5. 1. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra then V is a representation of $\mathfrak{g}$ via the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$. This corresponds to the action

$$x.v = x(v) \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

This is then called the *natural representation* of $\mathfrak{g}$.

2. If $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$ is a Lie subalgebra then $\mathfrak{g}$ acts on k^n via

$$x.v = x \cdot v \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in k^n,$$

which correspondings to the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(k^n), x \mapsto (v \mapsto x \cdot v)$. This is then called the *natural representation* of $\mathfrak{g}$.

3. Let $\mathfrak{g} \coloneqq \mathfrak{sl}_2(k)$ for any field k. Then $k[x, y]$ becomes a representation of $\mathfrak{g}$ via the homomorphism of Lie algebras $\rho \colon \mathfrak{g} \to \mathfrak{gl}(k[x, y])$ given by

$$\rho(e) = y \frac{\mathrm{d}}{\mathrm{d}x}, \quad \rho(h) = y \frac{\mathrm{d}}{\mathrm{d}y} - x \frac{\mathrm{d}}{\mathrm{d}x}, \quad \rho(f) = x \frac{\mathrm{d}}{\mathrm{d}y},$$

where x and y also denote the multiplication with the respective variable and (e, h, f) denotes the standard basis of $\mathfrak{sl}_2(k)$. To see that this is a homomorphism of representations notice that for all $n, m \geq 0$

$$\begin{aligned} e.(x^n y^m) &= n x^{n-1} y^{m+1}, \\ h.(x^n y^m) &= (m - n) x^n y^m, \\ f.(x^n y^m) &= m x^{n+1} y^{m-1}, \end{aligned}$$

where we write $x^\nu = 0$ and $y^\mu = 0$ for every $\nu, \mu < 0$. From this it follows that for all $n, m \geq 0$

$$\begin{aligned} e.f.(x^n y^m) - f.e.(x^n y^m) &= (n+1)m x^n y^m - n(m+1) x^n y^m \\ &= (m-n) x^n y^m = h.m = [e,f].(x^n y^m) \end{aligned}$$

as well as

$$\begin{aligned} h.e.(x^n y^m) - e.h.(x^n y^m) &= n(m-n+2) x^{n-1} y^{m+1} - n(m-n) x^{n-1} y^{m+1} \\ &= 2x^{n-1} y^{m+1} = 2e.(x^n y^m) = [h,e].(x^n y^m) \end{aligned}$$

and

$$\begin{aligned} h.f.(x^n y^m) - f.h.(x^n y^m) &= m(m-n-2) x^{n+1} y^{m-1} - m(m-n) x^{n+1} y^{m-1} \\ &= -2x^{n+1} y^{m-1} = -2f.(x^n y^{m-1}) = [h,f].(x^n y^{m-1}). \end{aligned}$$

4. Let $\mathfrak{g} \coloneqq \mathfrak{sl}_2(k)$ for any field k. Then the polynomial ring in one variable $k[x]$ is a representation of $\mathfrak{g}$ via the action the homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(k[x])$ with

$$\rho(e) = \frac{\mathrm{d}}{\mathrm{d}x}, \quad \rho(h) = -2x\frac{\mathrm{d}}{\mathrm{d}x}, \quad \rho(f) = -\frac{\mathrm{d}}{\mathrm{d}x}.$$

Then $\mathfrak{g}$ acts on $k[x]$ via

$$e.x^n = nx^{n-1}, \quad h.x^n = -2nx^n, \quad f.x^n = nx^{n+1} \quad \text{for every } n \geq 0,$$

where we set $x^m \coloneqq 0$ for $m < 0$. To see that this is really a representation of $\mathfrak{g}$ notice that for every $n \geq 0$

$$e.f.x^n - f.e.x^n = -n(n+1)x^n + n(n-1)x^n = -2nx^n = h.x^n = [e,f].x^n$$

as well as

$$h.e.x^n - e.h.x^n = -2n(n-1)x^{n-1} + 2n^2 x^{n-1} = 2nx^{n-1} = 2e.x^n = [h,e].x^n$$

and

$$h.f.x^n - f.h.x^n = 2n(n+1)x^{n+1} - 2n^2 x^{n+1} = 2nx^{n+1} = -2f.x^n = [h,f].x^n.$$

5. If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a Lie algebra $\mathfrak{g}$ and $\phi\colon \mathfrak{g}' \to \mathfrak{g}$ a homomorphism of Lie algebras then via the composition $\rho \circ \phi\colon \mathfrak{g}' \to \mathfrak{gl}(V)$ the vector space V becomes a representation of $\mathfrak{g}$. This corresponds to the action

$$x.v = \rho(x).v = \rho(\phi(x))(v) \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

6. The map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto \mathrm{ad}(x)$ is a homomorphism of Lie algebras and hence a representation of $\mathfrak{g}$.

Definition 1.2.6. Let $\mathfrak{g}$ be a Lie algebra. Then

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto [x, \cdot]$$

is called the *adjoint representation* of $\mathfrak{g}$.

Remark 1.2.7. Together with Lemma 1.1.36 it follows that every Lie algebras $\mathfrak{g}$ acts on itself by derivations of itself via the adjoint representation. This is where much of the structure of Lie algebras comes from and why the Jacobi identity is of interest (I guess).

Proposition 1.2.8 (New representations from old ones)**.** *Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field k.*

1. *If $(V_i)_{i\in I}$ is a collection of representations of $\mathfrak{g}$ then $\bigoplus_{i\in I} V_i$ is a representation of $\mathfrak{g}$ via*
$$x.\sum_{i\in I} v_i = \sum_{i\in I} x.v_i$$
or every $x \in \mathfrak{g}$ and $v_i \in V_i$ for all $i \in I$, with $v_i = 0$ for all but finitely many $i \in I$.

2. *If V and W are representations of $\mathfrak{g}$ then $V \otimes W$ is a representation of $\mathfrak{g}$ via*
$$x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w) \quad \text{for every } x \in \mathfrak{g},\ v \in V \text{ and } w \in W.$$
More generally: If $V_1, \dots V_n$ are representations of $\mathfrak{g}$ then $V_1 \otimes \cdots \otimes V_n$ is a representation of $\mathfrak{g}$ via
$$x.(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (x.v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n.$$
for every $x \in \mathfrak{g}$ and $v_i \in V_i$ for every $i = 1, \dots, n$.

3. *If V, W are representations of V then $\mathrm{Hom}_k(V, W)$ is a representation of $\mathfrak{g}$ via*
$$(x.f)(v) = x.f(v) - f(x.v) \quad \text{for every } x \in \mathfrak{g},\ f \in \mathrm{Hom}(V, W) \text{ and } v \in V.$$

4. *By letting $\mathfrak{g}$ act trivially on k the dual $V^* = \mathrm{Hom}_k(V, k)$ becomes a representation of $\mathfrak{g}$ in the above way, i.e.*
$$(x.\varphi)(v) = -\varphi(x.v) \quad \text{for every } x \in \mathfrak{g},\ \varphi \in V^* \text{ and } v \in V.$$

Proof. 1. Let $x, y \in \mathfrak{g}$ and $v_i \in V_i$ for every $i \in I$. Then
$$x.y.\sum_{i=1}^{n} v_i - y.x.\sum_{i=1}^{n} v_i = \sum_{i=1}^{n}(x.y.v_i - y.x.v_i) = \sum_{i=1}^{n}[x, y].v_i = [x, y].\sum_{i=1}^{n} v_i.$$

2. Let $x, y \in \mathfrak{g}$ and $v_i \in V_i$ for every $i = 1, \dots, n$. For all $i, j, m = 1, \dots, n$ set
$$\tilde{w}_m^{ij} := \begin{cases} x.v_i & \text{if } i = m \neq j, \\ y.v_i & \text{if } i \neq m = j, \\ x.y.v_i & \text{if } i = m = j, \\ v_i & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{w}_m^{ij} := \begin{cases} x.v_i & \text{if } i = m \neq j, \\ y.v_i & \text{if } i \neq m = j, \\ y.x.v_i & \text{if } i = m = j, \\ v_i & \text{otherwise.} \end{cases}$$

In particular $\tilde{w}_m^{ij} = \hat{w}_m^{ij}$ holds for all $m = 1, \dots, n$ and $i \neq j$. Therefore

$$\begin{aligned}
&x.y.(v_1 \otimes \cdots \otimes v_n) - y.x.(v_1 \otimes \cdots \otimes v_n) \\
&= \sum_{i,j=1}^{n} \tilde{w}_1^{ij} \otimes \cdots \otimes \tilde{w}_m^{ij} - \sum_{i,j=1}^{n} \hat{w}_1^{ij} \otimes \cdots \otimes \hat{w}_m^{ij} \\
&= \sum_{i=1}^{n} (\tilde{w}_1^{ii} \otimes \cdots \otimes \tilde{w}_m^{ii} - \hat{w}_1^{ii} \otimes \cdots \otimes \hat{w}_m^{ii}) \\
&= \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes (x.y.v_i) \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes (y.x.v_i) \otimes \cdots \otimes v_n) \\
&= \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (x.y.v_i - y.x.v_i) \otimes v_{i+1} \otimes \cdots v_n \\
&= \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes ([x,y].v_i) \otimes v_{i+1} \otimes \cdots v_n = [x,y].(v_1, \otimes \cdots \otimes v_n).
\end{aligned}$$

3. For all $x, y \in \mathfrak{g}$, $f \in \mathrm{Hom}(V, W)$ it follows for every $v \in V$ that

$$\begin{aligned}
(x.y.\varphi)(v) - (y.x.\varphi)(v) &= -(y.\varphi)(x.v) + (x.\varphi(y.v) = \varphi(y.x.v) - (\varphi(x.y.v) \\
&= -\varphi(x.y.v - y.x.v) = -\varphi([x,y].v) = ([x.y].\varphi)(v). \quad \square
\end{aligned}$$

1.2.2. Homomorphisms of representations

Definition 1.2.9. Let V und W be representations of a k-Lie algebra $\mathfrak{g}$. A k-linear map $f \colon V \to W$ is called a *homomorphism of representations* if

$$f(x.v) = x.f(v) \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

f is an *isomorphism of representations* if it is additionally bijective. If $\rho_V \colon \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho_W \colon \mathfrak{g} \to \mathfrak{gl}(W)$ are the corresponding Lie algebra homomomorphisms then f is a homomorphism of representations if and only if

$$f \circ \rho_V(x) = \rho_W(x) \circ f \quad \text{for every } x \in X.$$

The linear subspace of $\mathrm{Hom}_k(V, W)$ consisting of the homomorphisms of representations is denoted by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$, and for $V = W$ by $\mathrm{End}_{\mathfrak{g}}(V) \coloneqq \mathrm{Hom}_{\mathfrak{g}}(V, V)$.

Examples 1.2.10. Let $\mathfrak{g}$ be a Lie algebra over a field k.

1. If V is a representation of $\mathfrak{g}$ then $\mathrm{id}_V \colon \mathfrak{g} \to \mathfrak{g}$ is an automorphism of V as a representation of $\mathfrak{g}$.

2. If V_1, V_2 and V_3 are representations of $\mathfrak{g}$ and $f \colon V_1 \to V_2$ and $g \colon V_2 \to V_3$ are homomorphisms of representations then the composition $g \circ f \colon V_1 \to V_3$ is also a homomorphism of representations.

Remark 1.2.11. For a Lie algebra $\mathfrak{g}$ over a field k the representations of $\mathfrak{g}$ together with the homomorphisms of representations between them form a category.

Remark 1.2.12. If $f\colon V \to W$ is an isomorphism of representations of a Lie algebra $\mathfrak{g}$ then the k-linear map $f^{-1}\colon W \to V$ is also a homomorphism of representations (and therefore an isomorphism of representations) because

$$f^{-1}(x.v) = f^{-1}(x.f(f^{-1}(v))) = f^{-1}(f(x.f^{-1}(v))) = x.f^{-1}(v)$$

for every $x \in \mathfrak{g}$ and $v \in V$. It also follows for every $x \in \mathfrak{g}$ from $f\rho_V(x) = \rho_W(x)f$ that $\rho_V(x)f^{-1} = f^{-1}\rho_W(x)$.

Remark 1.2.13. Given two representations V and W of a Lie algebra $\mathfrak{g}$ a linear map $f\colon V \to W$ is a homomorphism of representations if and only if

$$x.f(v) - f(x.v) = 0 \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V,$$

which is equivalent to $x.f = 0$ for every $x \in \mathfrak{g}$ with respect to the induced action of $\mathfrak{g}$ on $\mathrm{Hom}_k(V, W)$. Hence the homomorphisms of representations are precisely the "invariant" linear maps under the action of $\mathfrak{g}$.

Proposition 1.2.14. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If V and W are finite dimensional representations of $\mathfrak{g}$ then the map*

   $$\Phi_1\colon V^* \otimes W \to \mathrm{Hom}_k(V, W), \quad \varphi \otimes w \mapsto (v \mapsto \varphi(v)w)$$

   is a homomorphism of representations. If V and W are both finite dimensional this is an isomorphism of vector spaces and thus already an isomorphism of representations.

2. *If $V_1, \dots, V_r$ and $W_1, \dots, W_s$ are representations of $\mathfrak{g}$ then the isomorphism of vector spaces*

   $$\begin{aligned} \Phi_2\colon (V_1 \otimes \cdots \otimes V_r) \otimes (W_1 \otimes \cdots \otimes W_s) &\longrightarrow V_1 \otimes \cdots \otimes V_r \otimes W_1 \otimes \cdots \otimes W_s, \\ (v_1 \otimes \cdots \otimes v_r) \otimes (w_1 \otimes \cdots \otimes w_s) &\longmapsto v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s \end{aligned}$$

   is already an isomorphism of representations.

3. *If V and W are representations of $\mathfrak{g}$ then the isomorphism of vector spaces*

   $$\Phi_3\colon V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w \otimes v$$

   is already an isomorphism of representations.

4. *If V_1, V_2 and W are representations of $\mathfrak{g}$ then the isomorphism of vector spaces*

   $$\begin{aligned} \Phi_4\colon (V_1 \otimes V_2) \otimes W &\to (V_1 \otimes W) \oplus (V_2 \otimes W), \\ (v_1 + v_2) \otimes w &\mapsto (v_1 \otimes w) + (v_2 \otimes w) \end{aligned}$$

   is already an isomorphism of representations.

5. *If $V_1, \dots, V_n$ are representations of $\mathfrak{g}$ and $\sigma \in S_n$ then the isomorphism of vector spaces*

$$\Phi_5\colon V_1 \otimes \cdots \otimes V_n \to V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}, \quad v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

is already an isomorphism of representations.

6. *If $V_1, \dots, V_n$ and $W_1, \dots, W_n$ are representations of $\mathfrak{g}$ and $f_i\colon V_i \to W_i$ for every $i = 1, \dots, n$ a homomorphism of representations it follows that the map*

$$\Phi_6\colon f_1 \otimes \cdots \otimes f_n\colon \bigotimes_{i=1}^n V_i \to \bigotimes_{i=1}^n W_i, \quad v_1 \otimes \cdots \otimes v_n \mapsto f(v_1) \otimes \cdots \otimes f(v_n)$$

is also a homomorphism of representations.

Proof. 1. For $x \in \mathfrak{g}$, $\varphi \in V^*$ and $w \in W$ it follows for every $v \in V$ that

$$\begin{aligned}\Phi_1(x.(\varphi \otimes w))(v) &= \Phi_1((x.\varphi) \otimes w + \varphi \otimes (x.w))(v) = (x.\varphi)(v)w + \varphi(v)x.w \\ &= \varphi(v)x.w - \varphi(x.v)w = x.(\varphi(v)w) - \varphi(x.v)w \\ &= x.\Phi_1(\varphi \otimes w)(v) - \Phi_1(\varphi \otimes w)(x.v) = (x.\Phi_1(\varphi \otimes w))(v).\end{aligned}$$

The other statements follow from similar calculations. □

1.2.3. Subrepresentations and irreducible representations

Definition 1.2.15. Let $\mathfrak{g}$ be a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$. A *subrepresentation* of V is a linear subpace $U \subseteq V$ such that $x.u \in U$ for every $x \in \mathfrak{g}$ and $u \in U$. Equivalently U is $\rho(x)$ invariant for every $x \in \mathfrak{g}$.

If $(U_i)_{i\in I}$ is a collection of subrepresentations of $\mathfrak{g}$ then V is called the *direct sum* of the U_i if $V = \bigoplus_{i\in I} U_i$ as vector spaces.

Examples 1.2.16. Let $\mathfrak{g}$ be a Lie algebra.

1. If V is a representation of $\mathfrak{g}$ then 0 and V itself are subrepresentations. These are also called the *trivial subrepresentations*.

2. If V is a representation and $(U_i)_{i\in I}$ a collection of subrepresentations $U_i \subseteq V_i$ then $\sum_{i\in I} U_i$ is also a subrepresentation of V.

3. Let V and W be representations of a Lie algebra $\mathfrak{g}$ and $\varphi\colon V \to W$ a homomorphism of representations. Then $\ker\varphi \subseteq V$ and $\operatorname{im}\varphi \subseteq W$ are subrepresentations.

4. The subrepresentations of the adjoint representation of $\mathfrak{g}$ are precisely the ideals in $\mathfrak{g}$.

5. If V is a representation of $\mathfrak{g}$ then

$$\mathfrak{g}V \coloneqq \{x.v \mid x \in \mathfrak{g}, v \in V\} \quad \text{and} \quad V^{\mathfrak{g}} \coloneqq \{v \in V \mid x.v = 0 \text{ for every } x \in \mathfrak{g}$$

are subrepresentations of $\mathfrak{g}$. Notice that if V and W are two representations of $\mathfrak{g}$ and $f\colon V \to W$ is a homomorphism of representations then $f(\mathfrak{g}V) \subseteq \mathfrak{g}W$ and $f(V^{\mathfrak{g}}) \subseteq W^{\mathfrak{g}}$. Hence the assignments $V \mapsto \mathfrak{g}V$ and $V \mapsto V^{\mathfrak{g}}$ define endofunctors of the category of representations of $\mathfrak{g}$ where homomorphisms are send to the corresponding restrictions.

6. Let V be a representation of $\mathfrak{g}$ and $f\colon V \to V$ a homomorphism of representations of $\mathfrak{g}$. Then for any $\lambda \in k$ the eigenspace

$$V_\lambda \coloneqq \{v \in V \mid f(v) = \lambda v\}$$

and the generalized eigenspace

$$V_{(\lambda)} \coloneqq \bigcup_{n\in\mathbb{N}} \ker(f - \lambda\,\mathrm{id}_V)^n = \{v \in V \mid (f - \lambda\,\mathrm{id}_V)^n(v) = 0 \text{ for some } n \in \mathbb{N}\}$$

are subrepresentations of V.

Definition 1.2.17. A representation V of a Lie algebra $\mathfrak{g}$ is called *irreducible* or *simple* if it has precisely two subrepresentations. Equivalently V is nonzero and has only the trivial subrepresentations.

The representation V is called *decomposable* if there exist non-trivial subrepresentations $U_1, U_2 \subseteq V$ with $V = U_1 \oplus U_2$. Otherwise V is called *indecomposable*.

The representation V is called *completely decomposible* or *semisimple* if there exists a decomposition $V = \bigoplus_{i\in I} U_i$ into irreducible subrepresentations $U_i \subseteq V$.

Remark 1.2.18. By definition every irreducible representation is also indecomposable. The converse does not hold. Irreducible representations are precisely the ones which are both indecomposable and completely reducible.

Example 1.2.19. 1. Every one-dimensional representation is irreducible.

2. The adjoint representation of a Lie algebra $\mathfrak{g}$ is irreducible if and only if $\mathfrak{g} \neq 0$ and $\mathfrak{g}$ has no ideals besides 0 and $\mathfrak{g}$ itself. So $\mathfrak{g}$ is either the onedimensional abelian Lie algebra or a simple Lie algebra.

Lemma 1.2.20 (Schur)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field k.*

1. *Let V and W be irreducible representations of $\mathfrak{g}$ and $\varphi\colon V \to W$ a homomorphism of representation. Then either $\varphi = 0$ or φ is an isomorphism of representations. In particular $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$ if $V \ncong W$, and $\mathrm{End}_{\mathfrak{g}}(V)$ is a skew field.*

2. *If k is algebraically closed and V a finite-dimensional irreducible representation of $\mathfrak{g}$ then every $\varphi \in \mathrm{End}_{\mathfrak{g}}(V)$ is given by multiplication with some $\lambda \in k$. In particular $\mathrm{End}_{\mathfrak{g}}(V) \cong k$ as rings.*

Proof. 1. From $V \neq 0$ and $W \neq 0$ it follows that φ cannot be zero and an isomorohpism at the same time. Suppose that $\varphi \neq 0$. Then $\operatorname{im}\varphi$ is a nonzero subrepresentation of W, hence $\operatorname{im}\varphi = W$ because W is irreducible. Similarly it follows that $\ker\varphi$ is a proper subrepresentation of V, hence $\ker\varphi = 0$ because V is irreducible. By combining these two observations it follows φ is an isomorphism.

2. As k is algebraically closed it follows that φ has an eigenvalue $\lambda \in k$. Therefore $\varphi - \lambda$ is an endomorphism of V as a representation of $\mathfrak{g}$ with nonzero kernel. Hence it is no isomorphism, so $\varphi - \lambda \operatorname{id}_V = 0$ because $\operatorname{End}_{\mathfrak{g}}(V)$ is a skew field. □

2. Slightly Advanced Basics

2.1. Nilpotent and solvable Lie algebras

2.1.1. Definition, examples and properties

Definition 2.1.1. Let A be an associative k-algebra. An element $a \in A$ is called *nilpotent* if $a^n = 0$ for some $n \geq 1$. Given a Lie algebra $\mathfrak{g}$ an element $x \in \mathfrak{g}$ is called ad-*nilpotent* if $\mathrm{ad}(x) \in \mathrm{End}_k(\mathfrak{g})$ is nilpotent.

Lemma 2.1.2. *If A is an associative k-algebra and $x \in A$ nilpotent then x is also* ad-*nilpotent.*

Proof. Let $\lambda_x \colon A \to A, a \mapsto xa$ and $\rho_x \colon A \to A, a \mapsto ax$. Because x is nilpotent both λ_x and ρ_x are nilpotent. Because A is associative λ_x and ρ_x commute. Hence $\mathrm{ad}(x) = \lambda_x - \rho_x$ is the sum of two commuting, nilpotent endomorphisms, and therefore also nilpotent. □

Definition 2.1.3. Let g be a Lie algebra. Define $\mathfrak{g}^0 := \mathfrak{g}$ and $\mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i]$ for all $i \in \mathbb{N}$. Then

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

is called the *central series* of $\mathfrak{g}$. Also define $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$ for all $i \in \mathbb{N}$. Then

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$$

is called the *derived series* of $\mathfrak{g}$. $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{g}^i = 0$ for some i and *solvable* if $g^{(i)} = 0$ for some i.

Examples 2.1.4. 1. Every nilpotent Lie algebra $\mathfrak{g}$ is also solvable because $\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^i$ for every $i \in \mathbb{N}$.

2. The upper triangular matrices $\mathfrak{t}_n(k)$ are solvable. But they are not nilpotent.

3. The strictly upper triangular matrices $\mathfrak{n}_n(k)$ not only solvable but also nilpotent.

4. If $n \geq 2$ then $\mathfrak{sl}_2(\mathbb{C})$ is simple and therefore $[\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$. Since $[\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$ it follows that $\mathfrak{gl}_n(\mathbb{C})$ is not solvable.

5. If $\mathfrak{g}$ is abelian then $\mathfrak{g}$ is nilpotent and therefore also solvable.

6. Every one-dimensional Lie algebra is abelian and therefore nilpotent and also solvable. The same goes for the two-dimensional abelian Lie algebra. The two-dimensional non-abelian Lie algebra $\mathfrak{g}$ has a basis x,y with $[x, y] = x$. Therefore $\mathfrak{g}$ is solvable but not nilpotent.

7. A *Heisenberg Lie algebra* consists of a real vector space with basis $P_1, \dots, P_n$, $Q_1, \dots, Q_n$, C together with the Lie bracket satisfying the following conditions:

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_i, C] = 0 \quad \text{and} \quad [P_i, Q_j] = \delta_{ij} C.$$

This defines a nilpotent Lie algebra.

Proposition 2.1.5. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If $\mathfrak{h}$ is a Lie algebra and $f\colon \mathfrak{g} \to \mathfrak{h}$ a Lie algebras homomorphism then $f(\mathfrak{g})^i = f(\mathfrak{g}^i)$ and $f(\mathfrak{g})^{(i)} = f(\mathfrak{g}^{(i)})$ for all $i \geq 0$.*
2. *If $\mathfrak{g}$ is nilpotent (resp. solvable) then any Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and any quotient of $\mathfrak{g}$ (by an ideal I) is nilpotent (resp. nilpotent).*
3. *If $I \trianglelefteq \mathfrak{g}$ with $I \subseteq Z(\mathfrak{g})$ and $\mathfrak{g}/I$ is nilpotent then $\mathfrak{g}$ is nilpotent.*
4. *If $\mathfrak{g} \neq 0$ is nilpotent then $Z(\mathfrak{g}) \neq 0$.*
5. *If $\mathfrak{g}$ is nilpotent and $x \in \mathfrak{g}$ then x is* ad*-nilpotent.*
6. *If $I \trianglelefteq \mathfrak{g}$ then I^i and $I^{(i)}$ are ideals inside $\mathfrak{g}$ for all $i \geq 0$.*

Proof. 1. It suficies to show that for any two subsets $X, Y \subseteq \mathfrak{g}$

$$f([X, Y]) = [f(X), f(Y)]$$

the statement then follows inductively. It holds because f is a Lie algebra homomorphism and therefore

$$\begin{aligned} f([X, Y]) &= f(\operatorname{span}_k\{[x, y] \mid x \in X, y \in Y\}) \\ &= \operatorname{span}_k\{f([x, y]) \mid x \in X, y \in Y\} \\ &= \operatorname{span}_k\{[f(x), f(y)] \mid x \in X, y \in Y\} \\ &= \operatorname{span}_k\{[x', y'] \mid x' \in f(X), y' \in f(Y)\} \\ &= [f(X), f(Y)]. \end{aligned}$$

2. The statement about subalgebras follows from $\mathfrak{h}^i \subseteq \mathfrak{g}^i$ and $\mathfrak{h}^{(i)} \subseteq \mathfrak{g}^{(i)}$ for all $i \in \mathbb{N}$. The statement about quotient follow by using the canonical projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$. Because π is a Lie algebra homomorphism it follows that

$$(\mathfrak{g}/I)^i = \pi(\mathfrak{g})^i = \pi(\mathfrak{g}^i) = 0$$

for i big enough. For solvable $\mathfrak{g}$ the corresponding statements follow in the same way.

3. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$ be the canonical projection. Because $\mathfrak{g}/I$ is nilpotent there exists some $i \geq 0$ with $(\mathfrak{g}/I)^i = 0$ and therefore

$$0 = (\mathfrak{g}/I)^i = \pi(\mathfrak{g})^i = \pi(\mathfrak{g}^i).$$

Thus $\mathfrak{g}^i \subseteq I \subseteq Z(G)$ and hence $\mathfrak{g}^{i+1} = 0$.

4. Let $i \in \mathbb{N}$ be minimal with $\mathfrak{g}^i \neq 0$ but $\mathfrak{g}^{i+1} = 0$. Then $\mathfrak{g}^i \subseteq Z(\mathfrak{g})$ and thus $Z(\mathfrak{g}) \neq 0$.
5. Since $\mathfrak{g}$ is nilpotent there exists some $i \in \mathbb{N}$ with $\mathfrak{g}^i = 0$. Then
$$(\mathrm{ad}(x))^i(\mathfrak{g}) \subseteq \mathfrak{g}^i = 0,$$
so $(\mathrm{ad}(x))^i = 0$.
6. This follows inductively by using that $[I, J]$ is an ideal inside $\mathfrak{g}$ for any $I, J \trianglelefteq \mathfrak{g}$.

□

Corollary 2.1.6. *If $I \trianglelefteq \mathfrak{g}$ is an ideal inside a Lie algebra $\mathfrak{g}$ then $\mathfrak{g}$ is solvable if and only if both I and $\mathfrak{g}/I$ are solvable.*

Proof. If $\mathfrak{g}$ is solvable then I and $\mathfrak{g}/I$ are also solvable by Proposition 2.1.5. Suppose that on the other hand both I and $\mathfrak{g}/I$ are solvable. Then there exists $i_1, i_2 \in \mathbb{N}$ with $(\mathfrak{g}/I)^{(i_1)} = 0$ and $I^{i_2} = 0$. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/I, x \mapsto x + I$ be the canical projection. Because
$$0 = (\mathfrak{g}/I)^{(i_1)} = \pi(\mathfrak{g})^{(i_1)} = \pi(\mathfrak{g}^{(i_1)})$$
it follows that $\mathfrak{g}^{(i_1)} \subseteq \ker \pi = I$. Thus
$$\mathfrak{g}^{(i_1+i_2)} = (\mathfrak{g}^{(i_1)})^{i_2} \subseteq I^{i_2} = 0,$$
which shows that $\mathfrak{g}$ is solvable. □

Remark 2.1.7. The analogeous statement about nilpotency does not necessarily hold. Take for example the two-dimensional non-abelian Lie algebra $\mathfrak{g}$, which has a basis x, y with $[x, y] = x$. Then the one-dimensional linear subspace $I := kx$ is an abelian ideal in $\mathfrak{g}$ and in particular nilpotent. The quotient $\mathfrak{g}/I$ is one-dimensional and therefore also nilpotent. But $\mathfrak{g}$ itself is not nilpotent.

Corollary 2.1.8. *Let $\mathfrak{g}$ be a Lie algebra and $I, J \trianglelefteq \mathfrak{g}$ two solvable ideal. Then $I + J$ is also solvable.*

Proof. Because J is solvable the same goes for $J/(I \cap J)$. Hence in the short exact sequence
$$0 \to I \to I + J \to (I + J)/I \to 0$$
both I and $(I+J)/I \cong J/(I \cap J)$ are solvable. Hence $I+J$ is solvable by Corollary 2.1.6.

□

Definition 2.1.9. Let $\mathfrak{g}$ be a finite dimenisonal Lie algebra. It follows from Corollary 2.1.8 that $\mathfrak{g}$ contains a unique maximal solvable ideal. This ideal is called the *radical* of $\mathfrak{g}$ and is denoted by $\mathrm{rad}\, \mathfrak{g}$.

Remark 2.1.10. If $\mathfrak{g}$ is a Lie algebra and $I, J \trianglelefteq \mathfrak{g}$ two nilpotent ideals, then it can be shown that the ideal $I + J$ is also a nilpotent. It follows that every finite dimensional Lie algebra $\mathfrak{g}$ has a unique maxmial nilpotent ideal, which is then called the *nilradical* of $\mathfrak{g}$.

2.1.2. Engel's theorem

From now on *all* fields over which we work will be assumed to be algebraically closed, unless otherwise specified.

If V is an n-dimensional vector space over k and $x \in \operatorname{End}_k(V)$ a nilpotent endomorphism then 0 is the only eigenvalue of x (and occurs with multiplicity n) (here it is used that k is algebraically closed). Hence there exists an eigenvector $v \in V$, $v \neq 0$ with $x(v) = 0$. The following proposition generalizes this observations for linear Lie algebras consisting of nilpotent endomorphisms.

Proposition 2.1.11. *Let $V \neq 0$ be a finite dimensional vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra such that every $x \in \mathfrak{g}$ in nilpotent. Then there exists $v \in V$ with $v \neq 0$ and $x(v) = 0$ for every $x \in \mathfrak{g}$, i.e. v is a common eigenvector of all $x \in \mathfrak{g}$ (all of which are nilpotent and thus have 0 as their only eigenvalue).*

Proof. The statement can be then shown by induction over $\dim \mathfrak{g}$. For $\dim \mathfrak{g} = 0$ the statement follows from $\mathfrak{g} = 0$ and for $\dim \mathfrak{g} = 1$ the statement follows as previously discussed from $\mathfrak{g} = kx$ with x being a nilpotent endomorphism of V.

So let $\dim \mathfrak{g} \geq 2$ and suppose that the statement holds for all smaller dimensions. Let $I \subseteq \mathfrak{g}$ be a maximal proper Lie subalgebra (such a subalgebra exists because it is precisely one of maximal dimension strictly smaller than n). Any $x \in \mathfrak{g}$ with $x \neq 0$ spans a one-dimensional subalgebra kx of $\mathfrak{g}$; because $\dim \mathfrak{g} \geq 2$ it is a proper one. This show that $I \neq 0$. It turns out that I is already in ideal in $\mathfrak{g}$:

By assumption $\mathfrak{g}$ consists of nilpotent endomorphisms and therefore of ad-nilpotent elemenents. In particular every $x \in I$ acts nilpotent on $\mathfrak{g}$ via $\operatorname{ad}(x)$ with I being an $\operatorname{ad}(x)$-invariant linear subspace. Therefore every $x \in I$ acts on the quotient vector space $\mathfrak{g}/I$ by an induced nilpotent endomorphism

$$\overline{\operatorname{ad}}(x)\colon \mathfrak{g}/I \to \mathfrak{g}/I, \quad y + I \mapsto \operatorname{ad}(x)(y) + I = [x, y] + I.$$

As the map $\overline{\operatorname{ad}}\colon I \to \mathfrak{gl}(\mathfrak{g}/I)$ is an homomorphism of Lie algebras (because ad is) the image $\{\overline{\operatorname{ad}}(x) \mid x \in I\} \subseteq \mathfrak{gl}(\mathfrak{g}/I)$ is an Lie subalgebra, consisting of nilpotent endomorphisms. From $I \neq 0$ it follows that $\dim \mathfrak{g}/I < \dim \mathfrak{g}$, so by induction assumption there exists some $y \in \mathfrak{g}$ with $\overline{\operatorname{ad}}(x)(y + I) = 0$ for every $x \in I$ and $y + I \neq 0 + I$. Hence $[x, y] \in I$ for every $x \in I$ but $y \neq I$. Hence $y \in N_{\mathfrak{g}}(I)$ with $y \neq I$, so I is properly contained in its normalizer. As I is a maximal proper subalgebra of $\mathfrak{g}$ it follows that $N_{\mathfrak{g}}(I) = \mathfrak{g}$, so I is an ideal. It is even one of codimension 1:

If I had not codimension 1 then $\dim \mathfrak{g}/I > 1$. Then $\mathfrak{g}/I$ contains a one-dimensional proper subalgebra L (as seen above), and the preimage $\pi^{-1}(J)$ under the canonical projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$ is then a proper subalgebra of $\mathfrak{g}$ properly containing I, which contradicts the maximality of I. Hence I has codimension 1.

As $I \subseteq \mathfrak{g}$ has codimension 1 there exists some $y \in \mathfrak{g}$ with $g = I \oplus ky$ (as vector spaces). Because $\dim I < \dim \mathfrak{g}$ it follows from the induction assumption that

$$U := \{v \in V \mid x(v) = 0 \text{ for every } x \in I\} = \bigcap_{x \in I} \ker x$$

is a nonzero linear subspace of V. It sufficies to show that U is y-invariant: Then there exists some eigenvector $u \in U$ of y for which necessarily $y(u) = 0$. If $u \in U$ then $[x, y] \in I$ for every $x \in I$ because $I \trianglelefteq \mathfrak{g}$ and therefore

$$x(y(u)) = [x, y](u) - y(x(u)) = 0 - y(0) = 0.$$

Hence $y(u) \in U$, so U is y-invariant. □

Proposition 2.1.12. *Let V be a finite dimensional vector space with $n = \dim V$ and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra. Then the following are equivalent:*

1. *$\mathfrak{g}$ consists of nilpotent endomorphisms.*
2. *There exists a complete flag of V*

$$V = V_n \supsetneq V_{n-1} \supsetneq V_{n-2} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = 0,$$

   with $x(V_i) \subseteq V_{i-1}$ for every $i = 1, \ldots, n$.

3. *There exists a basis of V with respect to which every $x \in \mathfrak{g}$ is represented by an strictly upper triangular matrix.*

Proof. The implication 1 ⇒ 2 can be shown by induction over $\dim V$. For $\dim V = 1$ set $V_0 := 0$ and $V_1 := V$. By assumption every $x \in \mathfrak{g}$ acts nilpotent on V, so $x(V) = 0$ because V is one-dimensional. Thus $V = V_1 \supsetneq V_0 = 0$ is a complete flag for V satisfying the condititons.

Now let $\dim V = n \geq 2$ and suppose the statement holds for all smaller dimensions. Let $v \in V$, $v \neq 0$ with $x(v) = 0$ for every $x \in \mathfrak{g}$ and $W := V/kv$. Every $x \in \mathfrak{g}$ induces an endomorphism

$$\overline{x}\colon W \to W, \quad v + kv \mapsto x(v) + kv.$$

By induction assumption exists a complete flag

$$W = W_{n-1} \supsetneq W_{n-2} \supsetneq W_{n-3} \supsetneq \cdots \supsetneq W_1 \supsetneq W_0 = 0$$

with $\overline{x}(W_i) \subseteq W_{i-1}$ for every $x \in \mathfrak{g}$ and $i = 1, \ldots, n-1$. By setting $V_i := \pi^{-1}(W_{i-1})$ for every $i = 1, \ldots, n$ and $V_0 = 0$ it follows that

$$V = V_n \supsetneq V_{n-1} \supsetneq V_{n-2} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = 0,$$

is a complete flag of V. On the one hand $x(V_1) = x(kv) = 0 = V_0$ for every $x \in \mathfrak{g}$ and on the other hand

$$\pi(x(V_i)) = \overline{x}(\pi(V_i)) = \overline{x}(W_{i-1}) \subseteq W_{i-2}$$

and therefore $x(V_i) \subseteq \pi^{-1}(W_{i-2}) = V_{i-1}$ for every $i = 2, \ldots, n$.

The implications 2 ⇒ 3 and 3 ⇒ 1 are basic facts from linear algebra. □

Theorem 2.1.13 (Engel)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if all its elements are* ad*-nilpotent.*

Proof. If $\mathfrak{g}$ is nilpotent then there exists some $i \in \mathbb{N}$ with $\mathfrak{g}^i = 0$, from which is follows from $\operatorname{ad}(x)^i(y) \in \mathfrak{g}^i$ for every $x, y \in \mathfrak{g}$ that $\operatorname{ad}(x)^i = 0$ for every $x \in \mathfrak{g}$, hence every $x \in \mathfrak{g}$ is ad-nilpotent.

On the other hand suppose that $\mathfrak{g}$ consists of ad-nilpotent elemenents. If $\mathfrak{g} = Z(\mathfrak{g})$ then $\mathfrak{g}$ is abelian and hence nilpotent, so it suffices to show the statement under the additional assumption that $Z(\mathfrak{g}) \subsetneq \mathfrak{g}$. Because $\mathfrak{g}/Z(\mathfrak{g}) \cong \operatorname{ad}\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ consisting of nilpotent Elemenents it follows from Proposition 2.1.12 that $\mathfrak{g}/Z(\mathfrak{g})$ is isomorphic to a Lie subalgebra of $\mathfrak{n}_n(k)$ for $n = \dim \mathfrak{g}/Z(\mathfrak{g}) \geq 1$. Because $\mathfrak{n}_n(k)$ is nilpotent the same goes for $\mathfrak{g}/Z(\mathfrak{g})$ as seen in Proposition 2.1.5. □

Remark 2.1.14. It is not true that every nilpotent Lie-subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ with V being a finite dimensional vector space is represented by upper triangular matrices with respect to some basis of V. For example the onedimensional subalgebra $k \operatorname{id}_V \subseteq \mathfrak{gl}(V)$ is abelian and hence nilpotent, but with respect to every basis of V represented by $kI \subseteq \mathfrak{gl}_n(k)$.

Corollary 2.1.15. *Let $\mathfrak{g}$ be a finite dimensional nilpotent k-Lie algebra. If $I \trianglelefteq \mathfrak{g}$ is a non-zero ideal then $I \cap Z(\mathfrak{g}) = 0$.*

Proof. $\mathfrak{g}$ acts on itself via the adjoint representation and by Engel's Theorem $\operatorname{ad}(x)$ is nilpotent for every $x \in \mathfrak{g}$. Hence $\{\operatorname{ad}(x)|_I \mid x \in \mathfrak{g}\} \subseteq \mathfrak{gl}(I)$ is a Lie subalgebra consisting of nilpotent endomorphisms. By Proposition 2.1.11 there exists some non-zero $y \in I$ with $\operatorname{ad}(x)|_I(y) = 0$ for every $y \in \mathfrak{g}$, i.e. $[\mathfrak{g}, y] = 0$. Then $y \in Z(\mathfrak{g}) \cap I$. □

2.1.3. Lie's theorem

From now on we will not only require *every* field k we work with to be algebraically closed, but also to be of characteristic 0. Unless otherwise stated this holds up to the last page (page 89) of this text. In particular all Lie algebras and vector spaces will be assumed to have such a field as their ground field, even if not explicitely stated.

Definition 2.1.16. Let V be a representation of a Lie algebra $\mathfrak{g}$. For $\lambda \in \mathfrak{g}^*$ the linear subspace

$$V_\lambda := \{v \in V \mid x.v = \lambda x \text{ for every } x \in \mathfrak{g}\}$$

is called the $\mathfrak{g}$-*weight space* of V with respect to λ. An element $\lambda \in \mathfrak{g}^*$ is called a *weight* of V of $V_\lambda \neq 0$.

Lemma 2.1.17 (Invariance Lemma)**.** *Let V be a finite dimensional representation of a Lie algebra $\mathfrak{g}$ and $I \trianglelefteq \mathfrak{g}$ an ideal. Then V is also a representation of I by restriction of the action of $\mathfrak{g}$ on V to I. For $\lambda \in I^*$ let V_λ be the I-weight space of V with respect to λ. Then V_λ is already a subrepresentation of $\mathfrak{g}$.*

Proof. For $v \in V$ and $x_1, \ldots, x_n \in \mathfrak{g}$ we will write

$$x_1 \cdots x_n v := x_1.(\ldots.(x_n.v)).$$

The case $V_\lambda = 0$ is welll understood, so for the rest of this proof we fix some $\lambda \in I^*$ with $V_\lambda \neq 0$.

That V_λ is a subrepresentation of $\mathfrak{g}$ means that $yv \in V_\lambda$ for every $y \in \mathfrak{g}$ and $v \in V_\lambda$, which is equivalent to $xyv = \lambda(x)yv$ for every $x \in I$, $y \in \mathfrak{g}$ and $v \in V_\lambda$. Because

$$xyv = [x,y]v + yxv = \lambda([x,y])v + \lambda(x)yv \quad \text{for every } x \in I,\ y \in \mathfrak{g} \text{ and } v \in V_\lambda$$

this is equivalent to $\lambda([x,y]) = 0$ for every $x \in I$ and $y \in \mathfrak{g}$.

Until further notice we fix some $y \in \mathfrak{g}$ and $v \in V_\lambda$ with $v \neq 0$. As V is finite dimensional there exists some maximal $n \geq 1$ such that v, yv, …, $y^n v$ are linearly independent. Let

$$W_i \coloneqq \operatorname{span}_k(v, yv, \dots, y^i v) \quad \text{for every } i = 0, \dots, n.$$

Because v, yv, …, $y^n v$, $y^{n+1}v$ are linearly dependent it follows that W_n is invariant under the action of y.

Claim. *The linear subspace W_i is for every $i = 0, \dots, n$ a subrepresentation of I. With respect to the basis w, yw, …, $y^i w$ of W_i the action of $x \in I$ is represented by an upper triangular matrix where every diagonal entry is $\lambda(x)$.*

Proof. The claim can be proven by induction over i. As $W_0 = kv$ with $xv = \lambda(x)v$ for every $x \in I$ the claim holds for $i = 0$. Suppose that $i < n$ and that the claim holds for W_0, …, W_i. If $x \in I$ then also $[x,y] \in I$ and therefore

$$xy^{i+1}v = \underbrace{[x,y]y^i v}_{\substack{\in W_i \\ \text{by induction}}} + yxy^i v \equiv yxy^i v \mod W_i.$$

By induction it is not only $xy^i v \in W_i$ but also $xy^i v + W_{i-1} = \lambda(x)y^i v + W_{i-1}$. Therefore

$$yxy^i v \equiv \lambda(x)y^{i+1}v \mod W_i.$$

This shows the claim for W_{i+1}. □

Let $x \in I$. As $[x,y] \in I$ it follows from the previous claim that the $(n+1)$-dimensional linear subspace W_n is invariant under the action of $[x,y]$, which is given by an endomorphism $\phi_{[x,y]} \in \operatorname{End}_k(W_n)$, and that $\phi_{[x,y]}$ is represented by an upper triangular matrix for which all diagonal entries are $\lambda([x,y])$. It follows that in particular

$$\operatorname{tr}\phi_{[x,y]} = (n+1)\lambda([x,y]) \tag{1}$$

On the other hand W_n is invariant under the action of both x (by the claim) and y, which act by endomorphisms $\phi_x, \phi_y \in \operatorname{End}_k(V)$. As V is a representation of the Lie algebra $\mathfrak{g}$ it follows that $\phi_{[x,y]} = [\phi_x, \phi_y]$ and thus $\operatorname{tr}\phi_{[x,y]} = 0$. Together with (1) it follows that $\lambda([x,y]) = 0$. □

As a generalization of Proposition 2.1.11 we have the following result about solvable linear Lie algebras.

Theorem 2.1.18 (Lie)**.** *Let $V \neq 0$ be a finite dimensional k-vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a solvable Lie subalgebra. Then there exists a common eigenvector for $\mathfrak{g}$, i.e. some $v \in V$, $v \neq 0$ with $x(v) \in kv$ for every $x \in \mathfrak{g}$.*

Proof. The statement can be shown by induction over $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 0$ then $\mathfrak{g} = 0$ and any $v \in V$ with $v \neq 0$ does the job. If $\dim \mathfrak{g} = 1$ then $\mathfrak{g} = kx$ for some $x \in \mathfrak{gl}(V)$ with $x \neq 0$. Then any eigenvector of x does the job (since k is assumed to be algebraically closed and $V \neq 0$ such an eigenvector does exist).

Suppose that $\dim \mathfrak{g} = n \geq 2$ and the statement holds for every smaller dimension. Similarly to the proof of Proposition 2.1.11 we will split this proof into four consecutive parts:

1. Finding an ideal $I \trianglelefteq \mathfrak{g}$ of codimension 1.
2. Finding common eigenvectors for I by induction.
3. Showing that $\mathfrak{g}$ stabilizes as nonzero subspace $U \subseteq V$ of such eigenvectors.
4. Writing $\mathfrak{g} = I \oplus ky$ (as vector spaces) and finding an eigenvector of y in U.

For the first step notice that $\mathfrak{g}$ is nonzero but solvable, so $[\mathfrak{g}, \mathfrak{g}] \trianglelefteq \mathfrak{g}$ is a proper ideal. Hence $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is nonzero abelian Lie algebra. Therefore there exists a linear subspace $J \subseteq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of codimension 1 and J is an ideal in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Hence the preimage $I = \pi^{-1}(J)$ for the canonical projection $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], x \mapsto x + [\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ of codimension 1.

For the second step notice that because $\mathfrak{g}$ is solvable the same goes for I. So by induction hypothesis there exists a common eigenvector for I. Hence there exists some $\lambda \in I^*$ with $U \coloneqq V_\lambda \neq 0$, where we view V as a representation of $\mathfrak{g}$ via $x.v = x(v)$ for every $x \in \mathfrak{g}$ and $v \in V$.

The third step follows directly from the invariance lemma.

For the fourth step let $y \in \mathfrak{g}$ with $\mathfrak{g} = I \oplus ky$ (as vector spaces). Since $\mathfrak{g}$ stabilizes U this holds in particular for y. As $U \neq 0$ it follows that there exists some eigenvector of y inside of U, which is then a common eigenvector for $\mathfrak{g}$. □

Remark 2.1.19. The proof for Lie's theorem given in the lecture is basically a less structured version of the one in [**Humphreys**], from where we took the idea of breaking down the proof into four steps to emphasize the similarities with the proof of Proposition 2.1.11 (which we found very useful for understanding the structure of the previous proof).

Proposition 2.1.20. *Let $V \neq 0$ be a finite dimensional k-vector space with $n = \dim V$ and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra. Then the following are equivalent:*

1. *$\mathfrak{g}$ is solvable.*
2. *$\mathfrak{g}$ stabilizes some complete flag of V, i.e. there exists a complete flag*

$$V = V_n \supsetneq V_{n-1} \supsetneq V_{n-2} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = 0,$$

with $x(V_i) \subseteq V_i$ for every $x \in \mathfrak{g}$ and $i = 0, \ldots, n$.

3. *There exists a basis of V with respect to which every $x \in \mathfrak{g}$ is represented by an upper triangular matrix. In particular $\mathfrak{g}$ is isomorphic to a Lie-subalgebra of $\mathfrak{t}_n(k)$ for $n = \dim V$.*

Corollary 2.1.21. *A finite-dimensional k-Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent then $[\mathfrak{g}, \mathfrak{g}]^{(i)} = 0$ for some $i \in \mathbb{N}$. Hence $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}]^{(i)} = 0$, so $\mathfrak{g}$ is solvable.

Suppose that $\mathfrak{g}$ is solvable. Then $\operatorname{ad} \mathfrak{g} \cong \mathfrak{g}/Z(\mathfrak{g})$ is a solvable subalgebra von $\mathfrak{gl}(\mathfrak{g})$. By Lie's theorem there exists a basis of $\mathfrak{g}$ with respect to which $\operatorname{ad} x$ is represented by an upper triangular matrix for each $x \in \mathfrak{g}$. As ad is an homomorphism of Lie algebras it follows that with respect to this basis $\operatorname{ad}(x)$ is represented by a strictly upper triangular matrix for every $x \in [\mathfrak{g}, \mathfrak{g}]$. Hence every $x \in [\mathfrak{g}, \mathfrak{g}]$ is ad-nilpotent, and therefore also $\operatorname{ad}_{[\mathfrak{g},\mathfrak{g}]}$-nilpotent. By Engel's theorem $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

Corollary 2.1.22. *Every irreducible representation of a solvable Lie algebra $\mathfrak{g}$ over k is onedimensional.*

Proof. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{g}$, and therefore in particular $V \neq 0$. Then $\operatorname{im} \rho \subseteq \mathfrak{gl}(V)$ is also solvable and by Lie's theorem there exists a common eigenvector $v \in V$, $v \neq 0$ for $\operatorname{im} \rho$. Because $x.v = \rho(x)(v) \in kv$ for every $x \in \mathfrak{g}$ it follows that the onedimensional linear subspace $kv \subseteq V$ is a nonzero subrepresentation of V. Because V is irreducible it follows that $V = kv$. □

Remark 2.1.23. Corollary 2.1.22 is actually equivalent to Lie's theorem: If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra with then V is a representation of $\mathfrak{g}$ via $x.v = x(v)$ for every $x \in \mathfrak{g}$ and $v \in V$. If $V \neq 0$ is finite dimensional then V contains an irreducible subrepresentation U of $\mathfrak{g}$ (simply take some nonzero subrepresentation of minimal dimension.) If $\mathfrak{g}$ additionally is solvable then by Corollary 2.1.22 the irreducible subrepresentation U is onedimensional, hence of the form $U = kv$ for some $v \in V$ with $v \neq 0$. From the definition of the action of $\mathfrak{g}$ on V it follows that v is common eigenvector of $\mathfrak{g}$.

As a consequence of this Lie's theorem as formulated in Theorem 2.1.18 was called "Lie's theorem – concrete form" in the lecture while Corollary 2.1.22 was stated as "Lie's theorem – abstract version".

Remark 2.1.24. Corollary 2.1.22 does not hold for general fields k, even if algebraically closed. To see this let k be an algebraically closed field with $\operatorname{char} k = 2$ and $\mathfrak{g} \coloneqq \mathfrak{sl}_2(k)$. Then $\mathfrak{g}$ has a basis (e, h, f) where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with

$$[h, e] = [h, f] = 0 \quad \text{and} \quad [e, f] = h.$$

Hence $\mathfrak{g}$ is solvable. Let $V := k^2$ be the natural representation of $\mathfrak{g}$, i.e. $\mathfrak{g}$ acts on V by $x.v = x(v)$ for every $x \in \mathfrak{g}$ and $v \in V$. Then

$$e.\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{and} \quad f.\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix} \quad \text{for every} \begin{pmatrix} x \\ y \end{pmatrix} \in V.$$

It follows that if $U \subseteq V$ is a nonzero subrepresentation then U contains either e_1 or e_2, and therefore also the other one. Hence $U = V$, which shows that V is an irreducible representation of $\mathfrak{g}$.

2.2. The Killing form and Cartan's criterion

2.2.1. Associative bilinear forms and the Killing form

Definition 2.2.1. Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field k. A bilinear form $\beta\colon \mathfrak{g} \times \mathfrak{g} \to k$ is called *associative* if

$$\beta(x, [y, z]) = \beta([x, y], z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Remark 2.2.2. As $\mathfrak{g}$ acts on itself by the adjoint representation it also acts on $(\mathfrak{g} \otimes_k \mathfrak{g})^*$ as described in Proposition 1.2.8, i.e. for every $\beta \in (\mathfrak{g} \otimes_k \mathfrak{g})^*$ and all $x, y, z \in \mathfrak{g}$

$$\begin{aligned}(y.\beta)(x \otimes z) &= -\beta(y.(x \otimes z)) = -\beta((y.x) \otimes z + x \otimes (y.z)) \\ &= -\beta([y, x] \otimes z) - \beta(x \otimes [y, z]) = \beta([x, y] \otimes z) - \beta(x \otimes [y, z]).\end{aligned}$$

Identifying $(\mathfrak{g} \otimes_k \mathfrak{g})^*$ with the bilinear forms on $\mathfrak{g}$ via the universal property of the tensor product it follows that a bilinear form $\beta\colon \mathfrak{g} \times \mathfrak{g} \to k$ is associative if and only if

$$y.\beta = 0 \quad \text{for every } y \in \mathfrak{g}.$$

Because of this associative bilinear forms on $\mathfrak{g}$ are also called *invariant*.

Lemma 2.2.3. *Let $\mathfrak{g}$ be a Lie algebra and $\beta\colon \mathfrak{g} \to \mathfrak{g}^*$ a bilinear form. Then β is associative if and only if the maps*

$$\varphi\colon \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \beta(x, \cdot) \quad \textit{and} \quad \psi\colon \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \beta(\cdot, x)$$

are homomorphisms of representations of $\mathfrak{g}$ (where $\mathfrak{g}$ acts on itself by the adjoint representation and therefore on $\mathfrak{g}^*$ as described in Proposition 1.2.8).

Proof. For all $x, y, z \in \mathfrak{g}$ the equalities

$$(x.\varphi(y))(z) = -\varphi(y)(x.z) = -\beta(y, x.z) = -\beta(y, [x, z])$$

and

$$\varphi(x.y)(z) = \beta(x.y, z) = \beta([x, y], z) = -\beta([y, x], z)$$

hold. Hence the associativity of β is equivalent to the equality $x.\varphi(y) = \varphi(x.y)$ holding for all $x, y \in \mathfrak{g}$. It can be shown similarly that the associativity of β is equivalent to ψ being a homomorphism of representations. □

Corollary 2.2.4. *If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\beta\colon \mathfrak{g}\times\mathfrak{g}\to k$ an associative and non-degenerate bilinear form then*

$$\varphi\colon \mathfrak{g}\to\mathfrak{g}^*, \quad x\mapsto \beta(x,\cdot) \quad \textit{and} \quad \psi\colon \mathfrak{g}\to\mathfrak{g}^*, \quad x\mapsto\beta(\cdot,x)$$

are isomorphisms of representations.

Definition 2.2.5. Let V be a vector space and $\beta\colon \mathfrak{g}\times\mathfrak{g}\to k$ a symmetric bilinear form. Then

$$\operatorname{rad}\beta \coloneqq \{x\in V \mid \beta(x,y)=0 \text{ for every } y\in V\}$$

is the *radical* of β.

Lemma 2.2.6. *Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field k and $\beta\colon \mathfrak{g}\times\mathfrak{g}\to k$ a symmetric and associative bilinear form. For any ideal $I\trianglelefteq\mathfrak{g}$ the orthogonal complement*

$$I^\perp \coloneqq \{y\in\mathfrak{g}\mid \beta(x,y)=0 \textit{ for every } x\in I\}$$

is also an ideal in $\mathfrak{g}$. In particular $\operatorname{rad}\beta=\mathfrak{g}^\perp$ is an Ideal in $\mathfrak{g}$.

Proof. For every $x\in\mathfrak{g}$, $y\in I^\perp$ and $z\in I$ it follows that $[x,z]\in I$ and therefore

$$\beta([x,y],z) = -\beta([y,x],z) = -\beta(y,[x,z]) = 0. \qquad \square$$

Remark 2.2.7. The proof of Lemma 2.2.6 did not use that β is symmetric. This artficial restraint is only there to simplify the situation and notation (we do not need to distinguish between orthogonal complements from the left and from the right.) The main example of an associative bilinear form will be the Killing form, which is symmetric, so this assumption will pose no problems to us.

Definition 2.2.8. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an arbitrary field k. The *Killing form* of $\mathfrak{g}$ is the bilinear form

$$\kappa\colon \mathfrak{g}\times\mathfrak{g}\to k \quad \text{with} \quad \kappa(x,y)=\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) \quad \text{for all } x,y\in\mathfrak{g}.$$

Lemma 2.2.9. *The Killing form κ of an finite dimensional Lie algebra $\mathfrak{g}$ over an arbitrary field k is associative and symmetric.*

Proof. Recall from linear algebra that for any finite dimensional k-vector space V and all endomorphisms $f_1,\dots,f_n\in\operatorname{End}_k(V)$

$$\operatorname{tr}(f_1\cdots f_n)=\operatorname{tr}(f_2\cdots f_n f_1).$$

For all $x,y\in\mathfrak{g}$ it follows that

$$\kappa(x,y)=\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))=\operatorname{tr}(\operatorname{ad}(y)\operatorname{ad}(x))=\kappa(y,x),$$

so κ is symmetric. For all x, y, z it follows that

$$\begin{aligned}\kappa(x,[y,z]) &= \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}([y,z])) = \operatorname{tr}(\operatorname{ad}(x)[\operatorname{ad}(y),\operatorname{ad}(z)]) \\ &= \operatorname{tr}(\operatorname{ad}(x)(\operatorname{ad}(y)\operatorname{ad}(z) - \operatorname{ad}(z)\operatorname{ad}(y))) \\ &= \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)\operatorname{ad}(z)) - \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(z)\operatorname{ad}(y)) \\ &= \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)\operatorname{ad}(z)) - \operatorname{tr}(\operatorname{ad}(y)\operatorname{ad}(x)\operatorname{ad}(z)) \\ &= \operatorname{tr}((\operatorname{ad}(x)\operatorname{ad}(y) - \operatorname{ad}(y)\operatorname{ad}(x))\operatorname{ad}(z)) \\ &= \operatorname{tr}([\operatorname{ad}(x),\operatorname{ad}(y)]\operatorname{ad}(z)) = \operatorname{tr}(\operatorname{ad}([x,y])\operatorname{ad}(z)) = \kappa([x,y],z). \qquad \square\end{aligned}$$

Example 2.2.10. Let $\mathfrak{g} \coloneqq \mathfrak{gl}_n(k)$ for some arbitrary field k. Then

$$\kappa(x,y) = 2n\operatorname{tr}(xy) - 2(\operatorname{tr} x)(\operatorname{tr} y) =: \beta(x,y). \quad \text{for all } x, y \in \mathfrak{g}.$$

To see this let $(e_{ij})_{i,j=1,\dots,n}$ be the standard basis of $\mathfrak{gl}_n(k)$, i.e. the (i,j)-th entry of e_{ij} is 1, all other entries are 0 (hence $e_{ij}e_k = \delta_{jk}e_i$ for every $j = 1,\dots,n$ where $(e_1,\dots,e_n)$ is the standard basis of k^n). In particular

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \text{for all } i,j,k,l = 1,\dots,n. \tag{2}$$

To show that $\kappa = \beta$ is sufficies to show that

$$\kappa(e_{ij},e_{kl}) = \beta(e_{ij},e_{kl}) \quad \text{for all } i,j,k,l = 1,\dots,n,$$

as both κ and β are bilinear forms on $\mathfrak{g}$. For all $k,l,g,h = 1,\dots,n$ it follows from (2) that

$$\operatorname{ad}(e_{kl})(e_{gh}) = [e_{kl},e_{gh}] = e_{kl}e_{gh} - e_{gh}e_{kl} = \delta_{lg}e_{kh} - \delta_{kh}e_{gl}.$$

It follows that for all $i,j,k,l,g,h = 1,\dots,n$

$$\begin{aligned}\operatorname{ad}(e_{ij})\operatorname{ad}(e_{kl})(e_{gh}) &= \operatorname{ad}(e_{ij})(\delta_{lg}e_{kh} - \delta_{kh}e_{gl}) = \delta_{lg}\operatorname{ad}(e_{ij})(e_{kh}) - \delta_{kh}\operatorname{ad}(e_{ij})(e_{gl}) \\ &= \delta_{lg}(\delta_{jk}e_{ih} - \delta_{ih}e_{kj}) - \delta_{kh}(\delta_{jg}e_{il} - \delta_{il}e_{gj}) \\ &= \delta_{jk}\delta_{lg}e_{ih} - \delta_{ih}\delta_{lg}e_{kj} - \delta_{jg}\delta_{kh}e_{il} + \delta_{il}\delta_{kh}e_{gj}\end{aligned}$$

and the coefficient of e_{gh} in this expression is

$$a_{gh} = \delta_{jk}\delta_{lg}\delta_{ig} + \delta_{il}\delta_{kh}\delta_{jh} - \delta_{ih}\delta_{lg}\delta_{kg}\delta_{jh} - \delta_{jg}\delta_{kh}\delta_{ig}\delta_{hl}.$$

It follows that for all $i,j,k,l = 1,\dots,n$

$$\begin{aligned}\kappa(e_{ij},e_{kl}) &= \sum_{g,h=1}^{n}(\delta_{jk}\delta_{lg}\delta_{ig} + \delta_{il}\delta_{kh}\delta_{jh} - \delta_{ih}\delta_{lg}\delta_{kg}\delta_{jh} - \delta_{jg}\delta_{kh}\delta_{ig}\delta_{hl}) \\ &= \sum_{g,h=1}^{n}\delta_{jk}\delta_{lg}\delta_{ig} + \sum_{g,h=1}^{n}\delta_{il}\delta_{kh}\delta_{jh} - \sum_{g,h=1}^{n}\delta_{ih}\delta_{lg}\delta_{kg}\delta_{jh} - \sum_{g,h=1}^{n}\delta_{jg}\delta_{kh}\delta_{ig}\delta_{hl} \\ &= n\delta_{jk}\delta_{il} + n\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl} = 2n\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} \\ &= 2n\delta_{jk}(\operatorname{tr} e_{il}) - 2(\operatorname{tr} e_{ij})(\operatorname{tr} e_{kl}) = 2n\operatorname{tr}(e_{ij}e_{kl}) - 2(\operatorname{tr} e_{ij})(\operatorname{tr} e_{kl}) \\ &= \beta(e_{ij},e_{kl}).\end{aligned}$$

Remark 2.2.11. If $\mathfrak{g}$ is a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$ then the corresponding trace form ϕ_ρ is defined as

$$\phi_\rho(x, y) := \operatorname{tr}(\rho(x)\rho(y)) \quad \text{for all } x, y \in \mathfrak{g}.$$

Replacing ad with ρ in the proof of Lemma 2.2.9 shows that ϕ_ρ is an associative and symmetric bilinear form on $\mathfrak{g}$. The Killing form κ of $\mathfrak{g}$ is then just the special case $\kappa = \phi_{\mathrm{ad}}$.

Lemma 2.2.12. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k. Then for any ideal $I \subseteq \mathfrak{g}$ the Killing form κ_I is given by restriction of the Killing form $\kappa_{\mathfrak{g}}$ to I, i.e. $\kappa_I = \kappa_{\mathfrak{g}}|_{I\times I}$.*

Proof. Let $x, y \in I$. Then I is $\operatorname{ad}_{\mathfrak{g}}(x)$-invariant. Let $(x_1, \dots, x_r)$ be a basis of I and $(x_1, \dots, x_r, x_{r+1}, \dots, x_s)$ one of $\mathfrak{g}$. With respect to the basis $(x_1, \dots, x_r)$ of I the endomorphism $\operatorname{ad}_I(x)$ is represented by a matrix $A_x \in \mathrm{M}_r(k)$ and $\operatorname{ad}_I(y)$ is represented by a matrix $A_y \in \mathrm{M}_r(k)$. As $I \trianglelefteq \mathfrak{g}$ is in ideal it follows that $\operatorname{im}\operatorname{ad}_{\mathfrak{g}}(x) \subseteq \mathfrak{g}$ and $\operatorname{im}\operatorname{ad}_{\mathfrak{g}}(y) \subseteq \mathfrak{g}$, which is why with respect to the basis $(x_1, \dots, x_s)$ the endomorphism $\operatorname{ad}_{\mathfrak{g}}(x)$ and $\operatorname{ad}_{\mathfrak{g}}(y)$ are represented by a matrices

$$C_x = \begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_s(k) \quad \text{and} \quad C_y = \begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_s(k)$$

for some matrices $B_x, B_y \in \mathrm{M}_{s-r,r}(k)$. Hence

$$\begin{aligned} \kappa_{\mathfrak{g}}(x, y) = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x)\operatorname{ad}_{\mathfrak{g}}(y)) &= \operatorname{tr}\left(\begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix}\right) \\ &= \operatorname{tr}\begin{pmatrix} A_xA_y & A_xB_y \\ 0 & 0 \end{pmatrix} = \operatorname{tr}(A_xA_y) = \operatorname{tr}(\operatorname{ad}_I(x)\operatorname{ad}_I(y)) = \kappa_I(x, y). \end{aligned}$$

□

Example 2.2.13. Let $\mathfrak{g} := \mathfrak{sl}_n(k) = [\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)]$. As seen in example 2.2.10 the Killing form of $\mathfrak{gl}_n(k)$ is given by

$$\kappa_{\mathfrak{gl}_n(k)}(x, y) = 2n\operatorname{tr}(xy) - 2(\operatorname{tr} x)(\operatorname{tr} y) \quad \text{for all } x, y \in \mathfrak{gl}_n(k).$$

Because $\mathfrak{sl}_n(k) \trianglelefteq \mathfrak{gl}_n(k)$ it follows from Lemma 2.2.12 that the Killing form of $\mathfrak{sl}_n(k)$ is given by

$$\kappa_{\mathfrak{sl}_n(k)}(x, y) = 2n\operatorname{tr}(xy) \quad \text{for all } x, y \in \mathfrak{sl}_n(k).$$

In particular the Killing form of $\mathfrak{sl}_n(k)$ is just a multiple of the trace form.

Lemma 2.2.14. *Let $\mathfrak{g}$ be a Lie algebra and $I_1, I_2 \trianglelefteq \mathfrak{g}$ ideals with $\mathfrak{g} = I_1 \oplus I_2$. Then $I_1 \perp I_2$ with respect to the Killing form κ of $\mathfrak{g}$. In particular it follows that for all $x, y \in \mathfrak{g}$ with $x = x_1 + x_2$ and $y = y_1 + y_2$ with respect to $\mathfrak{g} = I_1 \oplus I_2$*

$$\kappa(x, y) = \kappa_{I_1}(x_1, y_1) + \kappa_{I_2}(x_2, y_2)$$

Proof. Because $[I_1, I_2] \subseteq I_1 \cap I_2 = 0$ it follows that for every $z_1 \in I_1$ and $z_2 \in I_2$

$$(\operatorname{ad}(z_1)\operatorname{ad}(z_2))(\mathfrak{g}) = \operatorname{ad}(z_1)(\operatorname{ad}(z_2)(\mathfrak{g})) \subseteq \operatorname{ad}(z_1)(I_2) = 0.$$

Therefore $\operatorname{ad}(z_1)\operatorname{ad}(z_2) = 0$ and in particular $\kappa(z_1, z_2) = \operatorname{tr}(\operatorname{ad}(z_1)\operatorname{ad}(z_2)) = 0$. From Lemma 2.2.12 it further follows that

$$\begin{aligned}\kappa(x, y) &= \kappa(x_1, y_1) + \kappa(x_1, y_2) + \kappa(x_2, y_1) + \kappa(x_2, y_2)\\ &= \kappa(x_1, y_1) + \kappa(x_2, y_2) = \kappa_{I_1}(x_1, y_1) + \kappa_{I_2}(x_2, y_2).\end{aligned}$$

□

2.2.2. The concrete Jordan decomposition

Definition 2.2.15. Let V be an n-dimensional k-vector space and $x \in \operatorname{End}_k(V)$ (resp. $y \in \operatorname{M}_n(y)$). Then x (resp. y) is called *semisimple* if it is diagonalizable.

Remark 2.2.16. An endomorphism $x \in \operatorname{End}_k(V)$ as above is semisimple if and only if every x-invariant subspace of V has a direct summand which is also x-invariant. (This depends on k being algebraically closed.)

Theorem 2.2.17. *Let V be a finite dimensional k-vector space and $x \in \operatorname{End}_k(V)$.*

1. *There exist unique $x_s, x_n \in \operatorname{End}_k(V)$ satisfying the following properties:*
   a) *$x = x_s + x_n$.*
   b) *x_s is semisimple and x_n is nilpotent.*
   c) *x_s and x_n commute.*
2. *x_s and x_n are Polynomials in x without constant term, i.e. there exist polynomials $P, Q \in k[T]$ such that $P(0) = Q(0) = 0$ and $x_s = P(x)$ and $x_n = P(x)$. In particular an endomorphism of V commutes with x if and only if it commutes with x_s and x_n.*
3. *If $A \subseteq B \subseteq V$ are linear subspaces with $x(B) \subseteq A$ then also $x_s(B) \subseteq A$ and $x_n(B) \subseteq A$.*

Proof. Let $\chi(T)$ be the characteristic polynomial of x with $\chi(T) = \prod_{i=1}^n (T - \lambda_i)^{m_i}$ where $\lambda_i \neq \lambda_j$ for $i \neq j$. By the chinese reminder theorem the map

$$\begin{aligned} k[T]/(\chi) &\longrightarrow \prod_{i=1}^n k[T]/((T-\lambda_i)^{m_i}),\\ F + (\chi) &\longmapsto (F + ((T-\lambda_1)^{m_1}), \dots, F + (((T-\lambda_n)^{m_n})) \end{aligned} \tag{3}$$

is surjective. Thus there exists some polynomial $P \in k[T]$ with

$$P(T) \equiv \lambda_i \mod (T - \lambda_i)^{m_i} \quad \text{for ever } i = 1, \dots, n. \tag{4}$$

We can also assume that $P(0) = 0$. If $\lambda_i = 0$ for some i then this follows directly from (4). Otherwise the polynomials $(T - \lambda_1)^{m_1}$, …, $(T - \lambda_n)^{m_n}$, T are pairwise coprime, so by replacing $\chi(T)$ with $\tilde{\chi}(T) \coloneqq \chi(T)T$ in (3) results in a polynomial $\tilde{P}$ which does not only satisfy (4) (with P replaced by $\tilde{P}$) but also $\tilde{P} \bmod T = 0$.

Now let $Q(T) \coloneqq T - P(T)$ as well as $x_s \coloneqq P(x)$ and $x_n \coloneqq Q(x)$. Then $x = x_s + x_n$ and x_s and x_n commute, as both are polynomials in x. For every $i = 1, \dots, n$ let

$$V_i \coloneqq \ker(x - \lambda_i)^{m_i}$$

be the generalized eigenspace of x with respect to the eigenvalue λ_i. Is is known from linear algebra that $V = \bigoplus_{i=1}^n V_i$.

It follows from (3) that for every $i = 1, \dots, n$ there exists some polynomial $P_i \in k[T]$ with

$$P(T) = \lambda_i + P_i(T)(T - \lambda_i)^{m_i},$$

from which follows for every $v \in V_i$ that

$$x_s(v) = (\lambda_i \operatorname{id}_V + P_i(x)(x - \lambda_i)^{m_i})(v) = \lambda_i v + P_i(x)(\underbrace{(x - \lambda_i)^{m_i}(v)}_{=0}) = \lambda_i v.$$

Hence V_i is x_s-invariant with $x_s|_{V_i} = \lambda_i \operatorname{id}_{V_i}$ for every $i = 1, \dots, n$. As $V = \bigoplus_{i=1}^n V_i$ this shows that x_s is semisimple and V_i is precisely the eigenspace of x_s to the eigenvalue λ_i.

To see that x_s is nilpotent notice that for every $i = 1, \dots, n$ and $v \in V_i$

$$x_n(v) = x(v) - x_s(v) = x(v) - \lambda_i(v) = (x - \lambda_i \operatorname{id}_V)(v).$$

Hence V_i is x_n-invariant with $x_n|_{V_i} = x - \lambda_i \operatorname{id}_{V_i}$ for every $i = 1, \dots, n$. By definition of V_i it follows that $x_n|_{V_i}$ is nilpotent for every $i = 1, \dots, n$. Because $V = \bigoplus_{i=1}^n V_i$ it follows that x_n is nilpotent.

This shows the existence of the claimed decomposition. For the uniqueness let $y_s, y_n \in \operatorname{End}_k(V)$ be any two endomorphisms with $x = y_s + y_n$ where y_s is semisimple, y_n is nilpotent and y_s and y_n commute. As y_s and y_n commute it follows that each of them commutes with $x = y_s + y_n$. Because x_s and x_n are polynomials in x it follows from this that y_s and y_n both commute with x_s and x_n. Hence x_s, x_n, y_s and y_n are pairwise commuting. In particular x_s and y_s are simultaneously diagonalizable which is why $x_s - y_s$ is also semisimple. As x_n and y_n commute and are both nilpotent it also follows that $y_n - x_n$ is nilpotent. But from $x_s + x_n = x = y_s + y_n$ it follows that

$$\underbrace{x_s - y_s}_{\text{semisimple}} = \underbrace{y_n - x_n}_{\text{nilpotent}}.$$

Hence $x_s - y_s = y_n - x_n = 0$.

All other statements of the theorem directly follow from the construction of x_s and x_n and the fact that they are polynomials without constant term in x. □

Remark 2.2.18. An analogeous statement of Theorem 2.2.17 can be shown for $\mathrm{M}_n(k)$ instead of $\operatorname{End}_k(V)$. More precisely: Every matrix $x \in \mathrm{M}_n(k)$ can be uniquely decomposed into $x = x_s + x_n$ such that x_s is semisimple, x_n is nilpotent and x_s and x_n

commute. Both x_s and x_n are polynomials without constant term in x, so any matrix in $\mathrm{M}_n(k)$ commutes with x if and only if it commutes with x_s and x_n. If $A \subseteq B \subseteq k^n$ are linear subspaces such that B is carried into A by left multiplication with x, then the same goes for x_s and x_n.

Definition 2.2.19. Let V be an n-dimensional vector space and $x \in \mathrm{End}_k(V)$ (resp. $y \in \mathrm{M}_n(k)$). Then the decomposition $x = x_s + x_n$ from Theorem 2.2.17 (resp. the decomposition $y = y_s + y_n$ from Remark 2.2.18) is called the *concrete Jordandecomposition* of x (resp. y). The element x_s (resp. y_s) is called the *(concrete) semisimple part* of x (resp. y) and the element x_n (resp. y_n) is called the *(concrete) nilpotent part* of x (resp. y).

Definition 2.2.20. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $x \in \mathfrak{g}$ is called ad-*semisimple* if $\mathrm{ad}(x)$ is a semisimple endomorphism of $\mathfrak{g}$.

Lemma 2.2.21. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space V (resp. $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$) and $x \in \mathfrak{g}$.*

1. *If x is semisimple then x is also* ad*-semisimple.*
2. *If x is nilpotent then x is also* ad*-nilpotent.*

Proof. We only show the case $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, the proof for the case $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$ being essentially the same.

1. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ consisting of eigenvectors of x, where e_i belongs to the eigenvalue $\lambda_i \in k$. Then for all $i, j = 1, \ldots, n$ let $E_{ij} \in \mathrm{End}_k(\mathfrak{g})$ be defined by
$$E_{ij}(e_k) = \delta_{jk} e_i \quad \text{for every } k = 1, \ldots, n.$$
Then $(E_{ij})_{i,j=1,\ldots,n}$ is a basis of $\mathrm{End}_k(\mathfrak{g})$. For all $i, j, k = 1, \ldots, n$ it follows that
$$\begin{aligned}[x, E_{ij}](e_k) &= (xE_{ij} - E_{ij}x)(e_k) = x(E_{ij}(e_k)) - E_{ij}(x(e_k)) \\ &= \delta_{jk} x(e_i) - \lambda_k E_{ij}(e_k) = \delta_{jk}\lambda_i e_i - \delta_{jk}\lambda_k e_i \\ &= (\lambda_i - \lambda_j)\delta_{jk} e_i = (\lambda_i - \lambda_j) E_{ij} e_k.\end{aligned}$$
It follows that
$$\mathrm{ad}_{\mathfrak{gl}(V)}(x)(E_{ij}) = [x, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \quad \text{for all } i, j = 1, \ldots, n,$$
so $\mathrm{ad}_{\mathfrak{gl}(V)}$ is semisimple and therefore also the restriction $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(x)|_{\mathfrak{g}}$.
2. In Lemma 2.1.2 it was already shown that $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$ is nilpotent. From this it follows that the restriction $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(x)|_{\mathfrak{g}}$ is also nilpotent. □

Corollary 2.2.22. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for a finite dimensional vector space V (resp. $\mathfrak{g} = \mathfrak{gl}_n(k)$). If $x \in \mathfrak{g}$ has the Jordan decomposition $x = x_s + x_n$ then $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan decomposition of $\mathrm{ad}(x)$.*

Proof. As x_s is semisimple the same goes for $\mathrm{ad}(x_s)$ and as x_n is nilpotent the same goes for $\mathrm{ad}(x_n)$, each following from Lemma 2.2.21. As x_s and x_n commute so do $\mathrm{ad}(x_n)$ and $\mathrm{ad}(x_s)$ because ad is a homomorphism of Lie algebras. □

2.2.3. Cartan's Criterion

Lemma 2.2.23. *Let V be a finite dimensional k-vector space. Let $A \subseteq B \subseteq \mathfrak{gl}(V)$ be linear subspaces and let*

$$T := \{z \in \mathfrak{gl}(V) \mid \mathrm{ad}(z)(B) \subseteq A\}.$$

If $x \in T$ and $\mathrm{tr}(xz) = 0$ for every $z \in T$ then x is nilpotent.

Proof. Let $x = x_s + x_n$ be the concrete Jordan decompositon of x. Then the concrete Jordan decomposition of $\mathrm{ad}(x)$ is given by $\mathrm{ad}(x) = \mathrm{ad}(x_s)+\mathrm{ad}(x_n)$ by Corollary 2.2.22. As $\mathrm{ad}(x)(B) \subseteq A$ it follows from the properties of the concrete Jordan decomposition (see Theorem 2.2.17) that also $\mathrm{ad}(x_s)(B) = \mathrm{ad}(x)_s(B) \subseteq A$. Hence $x_s \in T$.

Let $(v_1, \dots, v_n)$ be an ordered basis of V with respect to which x is in Jordan normal form. Then with respect to this basis x_s is diagonal and x_n is strictly upper triangular. Let $\lambda_i \in k$ with $x_s(v_i) = \lambda_i v_i$ for every $i = 1, \dots, n$ and set

$$E := \mathrm{span}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_n) \subseteq k.$$

To show that x is nilpotent it suffices to show that $x_s = 0$, which is equivalent to $\lambda_i = 0$ for every $i = 1, \dots, n$. As this is the same as $E = 0$ it is enough to show that $f = 0$ for every $\mathbb{Q}$-linear map $f\colon E \to \mathbb{Q}$. For the rest of the proof we fix such an f. Let $z\colon V \to V$ be defined by

$$z(v_i) := f(\lambda_i)v_i \quad \text{for every } i = 1, \dots, n.$$

Claim. *$z \in T$.*

Proof. For all $i, j = 1, \dots, n$ let $e_{ij} \in \mathfrak{gl}(V)$ with

$$e_{ij}(v_k) = \delta_{jk} v_i \quad \text{for every } k = 1, \dots, n.$$

Then $(e_{ij})_{i,j=1,\dots,n}$ is a k-basis of $\mathfrak{gl}(V)$. As already seen in the proof of Corollary 2.2.22 this is a basis of eigenvectors of $\mathrm{ad}(x_s)$ where e_{ij} is an eigenvector of $\mathrm{ad}(x_s)$ with respect to the eigenvalue $\lambda_i - \lambda_j$. Because $(v_1, \dots, v_n)$ is also a basis of V consisting of eigenvalues of z, where v_i belongs to the eigenvalue $f(\lambda_i)$, it follows in the same way, that e_{ij} is an eigenvector of $\mathrm{ad}(z)$ with respect to the eigenvalue

$$\mu_{ij} := f(\lambda_i) - f(\lambda_j) = f(\lambda_i - \lambda_j)$$

for all $i, j = 1, \dots, n$. In particular it follows that if e_{ij} and $e_{i'j'}$ have the same eigenvalue with respect to $\mathrm{ad}(x_s)$, i.e. if $\lambda_i - \lambda_j = \lambda_{i'} - \lambda_{j'}$ then the same goes holds with respect to $\mathrm{ad}(z)$. Hence if $y \in \mathfrak{gl}(V)$ is an eigenvector of $\mathrm{ad}(x_s)$ with respect to the eigenvalue λ then y is an eigenvector of $\mathrm{ad}(z)$ with respect to the eigenvalue $f(y)$.

As $\mathrm{ad}(x)(B) \subseteq A \subseteq B$ there exists a decomposition $B = A \oplus N$ into linear subspaces with A being $\mathrm{ad}(x)$-invariant and thus decomposing into $\mathrm{ad}(x)$-eigenspaces and $N \subseteq \ker \mathrm{ad}(x)$. Then by the previous observations it follows that A also decomposes into $\mathrm{ad}(z)$-eigenspaces and that $\mathrm{ad}(z)(N) = 0$. Hence $\mathrm{ad}(z)(B) \subseteq A$ and thus $z \in T$. □

Now $\operatorname{tr}(xz) = \operatorname{tr}(x_s z) + \operatorname{tr}(x_n z)$. As x_s and z are both diagonal with respect to the basis $(v_1, \ldots, v_n)$ it follows that $\operatorname{tr}(x_s z) = \sum_{i=1}^n f(\lambda_i)\lambda_i$, and as $x_n z$ is also stricly upper triangular it also follows that $\operatorname{tr}(x_n z) = 0$. Together with $z \in T$ this results in

$$0 = \operatorname{tr}(xz) = \sum_{i=1}^{n} f(\lambda_i)\lambda_i$$

Because $f(\lambda_i) \in \mathbb{Q}$ and $\lambda_i \in E$ for every $i = 1, \ldots, n$ applying f to this equation results in

$$0 = \sum_{i=1}^{n} f(\lambda_i)^2.$$

It follows that $f(\lambda_i) = 0$ for every $i = 1, \ldots, n$ and thus $f = 0$. □

Remark 2.2.24. The proof of Lemma 2.2.23 was not actually given in the lecture itself and proving it was an exercise on the third exercise sheet. The proof given above is the one I came up with based on some hints given on the exercise sheet. At some point it will proberly be merged with the proof given in [**Humphreys**].

In the lecture a proof was given for the special case $k = \mathbb{C}$. But since I have some trouble understanding the details it is not included here (yet).

Lemma 2.2.25 (Cartan's criterion for $\mathfrak{gl}(V)$)**.** *Let V be a finite dimensional k-vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra. Then $\mathfrak{g}$ is solvable if and only if* $\operatorname{tr}(xy) = 0$ *for every $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. Suppose that $\mathfrak{g}$ is solvable. Then by Lie's theorem there exists a basis of V with respect to which $\mathfrak{g}$ is represented by upper triangular matrices. Then $[\mathfrak{g}, \mathfrak{g}]$ is represented by strictly upper triangular matrices, which is why xy is also represented by a stricly upper triangular matrix for every $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$. Hence $\operatorname{tr}(xy) = 0$ for every $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.

Now suppose that $\operatorname{tr}(xy) = 0$ for every $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$. Set $A \coloneqq [\mathfrak{g}, \mathfrak{g}]$, $B \coloneqq \mathfrak{g}$ and

$$T \coloneqq \{x \in \mathfrak{gl}(V) \mid \operatorname{ad}(x)(B) \subseteq A\}.$$

Let $x \in [\mathfrak{g}, \mathfrak{g}] \subseteq T$ and $z \in T$. Then $[z, \mathfrak{g}] = [z, B] \subseteq A = [\mathfrak{g}, \mathfrak{g}]$. Writing x as $x = \sum_{i=1}^n [a_i, b_i]$ with $a_i, b_i \in \mathfrak{g}$ for every $i = 1, \ldots, n$ it follows that

$$\begin{aligned} \operatorname{tr}(xz) &= \sum_{i=1}^{n} \operatorname{tr}([a_i, b_i]z) = \sum_{i=1}^{n} \kappa_{\mathfrak{gl}(V)}([a_i, b_i], z) \\ &= \sum_{i=1}^{n} \kappa_{\mathfrak{gl}(V)}(a_i, [b_i, z]) = \sum_{i=1}^{n} \operatorname{tr}(a_i \underbrace{[b_i, z]}_{\in [\mathfrak{g}, \mathfrak{g}]}) = 0, \end{aligned}$$

where the last step uses the asumption. It follows from Lemma 2.2.23 that x is nilpotent. Because $[\mathfrak{g}, \mathfrak{g}]$ consists of nilpotent elements there exists a basis of V with respect to which $[\mathfrak{g}, \mathfrak{g}]$ is represented by stricly upper triangular matrices. Hence $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent und $\mathfrak{g}$ therefore solvable. □

Theorem 2.2.26 (Cartan's criterion for solvability)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}$ is solvable if and only if*

$$\kappa(x, y) = 0 \quad \textit{for every } x \in \mathfrak{g} \textit{ and } y \in [\mathfrak{g}, \mathfrak{g}].$$

Proof. Because $Z(\mathfrak{g})$ is a solvable ideal in $\mathfrak{g}$ it follows that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}/Z(\mathfrak{g}) \cong \operatorname{ad}\mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$ is solvable. By Cartan's criterion for $\mathfrak{gl}(\mathfrak{g})$ this is the case if and only if

$$\operatorname{tr}(xy) = 0 \quad \text{for every } x \in \operatorname{ad}\mathfrak{g} \text{ and } y \in [\operatorname{ad}(\mathfrak{g}), \operatorname{ad}(\mathfrak{g})].$$

Because $[\operatorname{ad}(\mathfrak{g}), \operatorname{ad}(\mathfrak{g})] = \operatorname{ad}([\mathfrak{g}, \mathfrak{g}])$ and $\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = \kappa(x, y)$ for all $x, y \in \mathfrak{g}$ this is equivalent to

$$\kappa(x, y) = 0 \quad \text{for every } x \in \mathfrak{g} \text{ and } y \in [\mathfrak{g}, \mathfrak{g}].$$ □

Corollary 2.2.27. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and κ the Killing form of $\mathfrak{g}$. Then $\operatorname{rad}\kappa$ is a solvable ideal of $\mathfrak{g}$. In particular $\operatorname{rad}\kappa \subseteq \operatorname{rad}\mathfrak{g}$.*

Proof. Lemma 2.2.6 already showed that $\operatorname{rad}\kappa$ is an ideal in $\mathfrak{g}$. From Lemma 2.2.12 and the definition of $\operatorname{rad}\kappa$ it follows that

$$\kappa_{\operatorname{rad}\mathfrak{g}}(x, y) = \kappa(x, y) = 0 \quad \text{for all } x, y \in \operatorname{rad}\kappa.$$

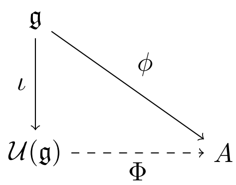
Hence by Cartan's criterion $\operatorname{rad}\kappa$ is solvable. □

2.3. The universal enveloping algebra

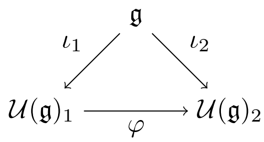
The following hold for this section alone: We fix an arbitrary field k and a Lie algebra $\mathfrak{g}$ over k. By a k-algebras we always mean an associative and unitary one, and homomorphisms of k-algebras have to respect the unit. The resulting category of k-algebras with homomorphisms of k-algebras between them will be denoted by k-**Alg**.

2.3.1. Definition, properties and construction

Definition 2.3.1. An *universal enveloping algebra* of $\mathfrak{g}$ is an k-algebra $\mathcal{U}(\mathfrak{g})$ together with a homomorphism of Lie algebras $\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that for every k-Algebra A and homomorphism of Lie algebras $\phi\colon \mathfrak{g} \to A$ there exists a unique homomorphism of k-algebras $\Phi\colon \mathcal{U}(\mathfrak{g}) \to A$ with $\phi = \Phi \circ \iota$, i.e. making the following diagram commute:



Remark 2.3.2. As always with universal objects any two enveloping algebras of $\mathcal{U}(\mathfrak{g})_1$ with $\iota_1\colon \mathfrak{g}_1 \to \mathcal{U}(\mathfrak{g})_1$ and $\mathcal{U}(\mathfrak{g})_2$ with $\iota_2\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})_2$ of $\mathfrak{g}$ are isomorphic, and there exists a unique isomorphism $\varphi\colon \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_2)$ with $\iota_2 = \varphi\colon \iota_1$, i.e. making the following diagram commute:



Hence we will talk about *the* universal enveloping algebra of $\mathfrak{g}$.

Proposition 2.3.3. *Let V be a vector space over k. Then there exists a bijection*

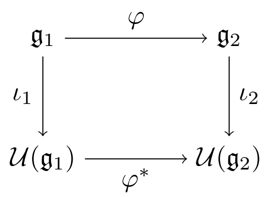
$$\left\{ \begin{array}{c} \textit{Representations of } \mathfrak{g} \\ \rho\colon \mathfrak{g} \to \mathfrak{gl}(V) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{U}(\mathfrak{g})\textit{-Modulstrukturen} \\ \theta\colon \mathcal{U}(\mathfrak{g}) \to \operatorname{End}_k(V) \end{array} \right\},$$
$$\rho \longmapsto \hat{\rho},$$
$$\theta|_{\mathfrak{g}} \longleftarrow \theta,$$

where $\hat{\rho}\colon \mathcal{U}(\mathfrak{g}) \to \operatorname{End}_k(V)$ is the k-algebra homomorphism induced by the homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ via the universial property of the universal enveloping algebra.

Proof. This is a direct consequence of the universal property of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. ☐

Remark 2.3.4. By Proposition 2.3.3 the category of representations of $\mathfrak{g}$ is isomorphic to the category of modules over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Remark 2.3.5. Given any two k-Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then any homomorphism of Lie algebras $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ induces a homomorphism of k-algebras $\phi^*\colon \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_2)$ via the following commutative diagram:



Hence the assignment $\mathfrak{g} \mapsto \mathcal{U}(g)$ of a Lie algebra to its universal eveloping algebra can be extended to a (covariant) functor $\mathcal{U}\colon k\text{-}\mathbf{Lie} \to k\text{-}\mathbf{Alg}$. It is by the universal property of the universal enveloping algebra left adjoint to the functor $k\text{-}\mathbf{Alg} \to k\text{-}\mathbf{Lie}$ which assignes each k-algebra its Lie algebra.

Lemma 2.3.6. *Let $T(\mathfrak{g}) = \bigoplus_{n\in\mathbb{N}} \mathfrak{g}^{\otimes n}$ be the tensor algebra and $\mathcal{I} \subseteq T(\mathfrak{g})$ the two-sided ideal generated by the element $x\otimes y - y\otimes x - [x,y]$ with $x, y \in \mathfrak{g}$. The the quotient $\mathcal{U}(\mathfrak{g}) \coloneqq T(\mathfrak{g})/\mathcal{I}$ together with the k-linear map*

$$\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g}), \quad x \mapsto x + \mathcal{I}$$

is an universal enveloping algebra of $\mathfrak{g}$.

Proof. $\mathcal{U}(\mathfrak{g})$ is a k-algebra by construction and ι is a homomorphism of Lie algebras since for all $x, y \in \mathfrak{g}$

$$\begin{aligned}[\iota(x), \iota(y)] &= [x+\mathcal{I}, y+\mathcal{I}] = (x+\mathcal{I})(y+\mathcal{I}) - (y+\mathcal{I})(x+\mathcal{I}) \\ &= (x\otimes y - y\otimes x) + \mathcal{I} = [x,y] + \mathcal{I} = \iota([x,y]).\end{aligned}$$

Given any k-algebra A and homomorphism of Lie algebras $\phi\colon \mathfrak{g} \to A$ it can be uniquely extended to a homomorphism of k-algebras $\hat{\phi}\colon T(\mathfrak{g}) \to A$ via

$$\hat{\phi}(x_1 \otimes \cdots \otimes x_n) = \phi(x_1)\cdots\phi(x_n) \quad \text{for all } n \geq 0 \text{ and } x_1, \dots, x_n \in \mathfrak{g}.$$

Because ϕ is not only k-linear but even a homomorphism of Lie algebras it follows that for all $x, y \in \mathfrak{g}$

$$\hat{\phi}(x\otimes y - y\otimes x) = \phi(x)\phi(y) - \phi(y)\phi(x) = [\phi(x), \phi(y)] = \phi([x,y]) = \hat{\phi}([x,y])$$

It follows that $\hat{\phi}(x) = 0$ for every $x \in \mathcal{I}$. Hence $\hat{\phi}$ factors through a unique homomorphism of k-algebras

$$\Phi\colon \mathcal{U}(\mathfrak{g}) \to A, \quad x_1 \otimes \cdots \otimes x_n + \mathcal{I} \mapsto \phi(x_1)\cdots\phi(x_n)$$

for all $n \geq 0$ and $x_1, \dots, x_n \in \mathfrak{g}$. For every $x \in \mathfrak{g}$ it follows that

$$(\Phi \circ \iota)(x) = \Phi(\iota(x)) = \Phi(x + \mathcal{I}) = \phi(x),$$

which is why $\phi = \Phi \circ \iota$. That Φ is the unique homomorphism of k-algebras with this properties follows from the uniqueness of $\hat{\phi}$. □

Corollary 2.3.7. *The homomorphism $\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective. As a k-algebra $\mathcal{U}(\mathfrak{g})$ is generated by $\iota(\mathfrak{g})$.*

Remark 2.3.8. We will always identify $\mathfrak{g}$ with its image under ι.

2.3.2. Poincaré–Birkhoff–Witt

2.3.2.1. Graded k-algebras

Definition 2.3.9. A *grading*, also called *gradation*, of a k-algebra A is a direct sum decomposition $A = \bigoplus_{i\in\mathbb{N}} A_i$ into linear subspaces such that

$$A_i A_j \subseteq A_{i+j} \quad \text{for all } i, j \in \mathbb{N}.$$

A *graded k-algebra* is a k-algebra A together with a grading $A = \bigoplus_{n\in\mathbb{N}} A_n$.

Remark 2.3.10. While a graded k-algebra is formally a pair $(A, (A_n)_{n\in\mathbb{N}})$ consisting of a k-algebra A and a grading $A = \bigoplus_{n\in\mathbb{N}} A_n$ we will often just call A a graded k-algebra without explicitily mentioning the grading. We also set $A_n := 0$ for every $n < 0$.

Remark 2.3.11. Given any semigroup $(S, \cdot)$ an S-*grading* of a k-algebra A is a decomposition $A = \bigoplus_{s\in S} A_s$ into linear subspaces such that $A_s A_t \subseteq A_{s\cdot t}$ for all $s, t \in S$. An S-graded k-algebra is a k-algebra A together with an S-grading $A = \bigoplus_{s\in S} A_s$. An graded k-algebra in the sense of Definition 2.3.9 is then the special case of an $\mathbb{N}$-graded k-algebra.

Lemma 2.3.12. *Let A be a graded k-algebra. Then $1 \in A_0$ and A_0 is a k-subalgebra.*

Proof. Let $1 = \sum_{i\in\mathbb{N}} e_i$ with respect to $A = \bigoplus_{n\in\mathbb{N}} A_n$. Then for any $j \in \mathbb{N}$ and $a \in A_j$

$$A_j \ni a = a \cdot 1 = a \left(\sum_{i\in\mathbb{N}} e_i \right) = \sum_{i\in\mathbb{N}} \underbrace{a e_i}_{\in A_{i+j}},$$

and it follows from the directness of the decomposition $A = \bigoplus_{n\in\mathbb{N}} A_n$ that $a = ae_0$. It follows that $ae_0 = a$ for every $a \in A$, hence e_0 is the unit of A.

That A_0 is a linear subspace which is closed under the multiplication follows from the definition of a graded k-algebra. As it contains the unit of A it is a k-subalgebra. □

Examples 2.3.13. 1. Any k-algebra A becomes a graded k-algebra by setting $A_0 := A$ and $A_i := 0$ for every $i > 1$.

2. The polynomial ring $A = k[x_1, \dots, x_n]$ is a graded k-algebra by setting

$$A_d := \operatorname{span}_k\{x_1^{p_1} \cdots x_n^{p_n} \mid p_1 + \cdots + p_n = d\} \quad \text{for every } d \in \mathbb{N},$$

i.e. A_d consists of the homogeneous polynomials of degree d.

3. Let V be a k-vector space. Then the tensor algebra $T(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$, the symmetric algebra $S(V) = \bigoplus_{n\in\mathbb{N}} S^n(V)$ and the exterior algebra $\Lambda(V) = \bigoplus_{n\in\mathbb{N}} \Lambda^n(V)$ carry the structure of a graded k-algebra via $T(V)_n := V^{\otimes n}$, $S(V)_n := S^n(V)$ and $\Lambda(V)_n := \Lambda^n(V)$ for every $n \in \mathbb{N}$.

Definition 2.3.14. Let A and B be graded k-algebras. A homomorphism of k-algebras $\varphi\colon A \to B$ is called a *homomorphism of graded k-algebras* if $\varphi(A_n) \subseteq B_n$ for every $n \in \mathbb{N}$, and the induced homomorphisms of vector spaces are denoted by $\varphi_n\colon A_n \to B_n$ for every $n \in \mathbb{N}$. An homomorphism of graded k-algebras is called an isomorphism if it is bijective.

Remark 2.3.15. If A is a graded k-algebra then id_A is a homomorphism of graded k-algebras, and if B and C are two other graded k-algebras and $\varphi\colon A \to B$ and $\psi\colon B \to C$ homomorphisms of graded k-algebras then so is $\psi\colon \varphi\colon A \to C$. Hence the graded k-algebras together with the homomorphisms of graded k-algebras between them form a category, which will be refered to by k-**Grad**.

Example 2.3.16. 1. For any vector space V the two maps

$$T(V) \to S(V), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$$

and

$$T(V) \to \Lambda(V), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \wedge \cdots \wedge x_n$$

are homomorphisms of graded k-algebras.

2. If V is a finite dimensional vector space with basis $x_1, \dots, x_n$ the the isomorphism of k-algebras

$$k[T_1, \dots, T_n] \to S(V), \quad T_i \mapsto x_i \quad \text{for every } i = 1, \dots, n$$

is already an isomorphism of graded k-algebras.

Definition 2.3.17. Let A be a graded k-algebra. A two-sided ideal $J \subseteq A$ is called *homogeneous* if $J = \bigoplus_{n\in\mathbb{N}}(J \cap A_n)$. Equivalently, given any $x \in J$ with the decomposition $x = \sum_{n\in\mathbb{N}} x_n$ with respect to $A = \bigoplus_{n\in\mathbb{N}} A_n$ it follows that $x_n \in J$ for every $n \in \mathbb{N}$.

Lemma 2.3.18. *Let A be a graded k-algebra and $J \subseteq A$ a two-sided homogeneous ideal. Then A/J is a graded k-algebra via $(A/J)_n = A_n/(J \cap A_n)$ for every $n \in \mathbb{N}$ and the canonical projection $\pi\colon A \to A/J, a \mapsto a + J$ is a homomorphism of graded k-algebras.*

Proof. This follows directly form the definition of a homogeneous ideal. □

Lemma 2.3.19. *Let A be a graded k-algebra, $J \subseteq A$ a two-sided ideal and call an element $x \in J$ homogeneous (in J) if $x_n \in J$ for every $n \in \mathbb{N}$ where $x = \sum_{n\in\mathbb{N}} x_n$ with respect to $A = \bigoplus_{n\in\mathbb{N}} A_n$. Then J is homogeneous if and only if J is generated by elements which are homogeneous in J.*

Proof. Let $I \coloneqq \{x \in J \mid x \text{ is homogeneous in } J\}$ be the linear subspace of elements which are homogeneous in J. If $x \in I$ then $x_n \in J$ for every $n \in \mathbb{N}$ where $x = \sum_{n\in\mathbb{N}} x_n$ with respect to $A = \bigoplus_{n\in\mathbb{N}} A_n$. Given $a \in A$ with $a = \sum_{m\in\mathbb{N}} a_m$ with respect to $A = \bigoplus_{m\in\mathbb{N}} A_m$ it follows that $a_m x_n \in J$ for all $m, n \in \mathbb{N}$ because J is left ideal, and therefore $ax = \sum_{m,n\in\mathbb{N}} a_m x_n \in J$. Hence I is a left ideal. In the same way it follows that I is also a right ideal and hence already a two-sided ideal in A.

The ideal J is homogeneous if and only if any of its elements is homogeneous in J, i.e. if $I = J$, from which the statement follows. □

Examples 2.3.20. 1. If A and B are graded k-algebras and $\varphi\colon A \to B$ a homomorphism of graded k-algebras then $\ker \varphi$ is a homogeneous ideal.

2. Let V be any vector space. The two-sided ideal I of $T(V)$ generated by the elements $x \otimes y - y \otimes x$ with $x, y \in V$ is a homogeneous ideal of $T(V)$. The same goes for the two-sided ideal J generated by the elements $x \otimes x$ with $x \in V$. The resulting (graded) quotient algebras are $S(V)$ and $\Lambda(V)$.

2.3.2.2. Filtered k-algebras

Definition 2.3.21. A *filtration* of a k-algebra A is an increasing sequence

$$A_{(0)} \subseteq A_{(1)} \subseteq A_{(2)} \subseteq \cdots \subseteq A$$

such that $A = \bigcup_{i\in\mathbb{N}} A_{(i)}$ and $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$ for all $i, j \in \mathbb{N}$, as well as $1 \in A_{(0)}$. A *filtered k-algebra* is a k-algebra A together with a filtration $A = \bigcup_{n\in\mathbb{N}} A_{(n)}$.

Remark 2.3.22. As for graded k-algebras we will refer to a filtered k-algebra A without explicitely mentioning the filtration. We also set $A_{(n)} := 0$ for every $n < 0$.

Definition 2.3.23. Let A and B be filtered k-algebras. A homomorphism of k-algebras $\varphi\colon A \to B$ is called a *homomorphism of filtered k-algebras* if $\varphi(A_{(n)}) \subseteq B_{(n)}$ for every $n \in \mathbb{N}$.

Remark 2.3.24. If A, B and C are filtered k-algebras then id_A is a homomorphism of filtered k-algebras and if $\varphi\colon A \to B$ and $\psi\colon B \to C$ are homomorphisms of filtered k-algebras then $\psi \circ \varphi$ is also a homomorphism of filtered k-algebras. It follows that filtered k-algebras together with homomorphisms of filtered k-algebras between them form a category, which will be refered to by k-**Filt**.

Examples 2.3.25. 1. Any graded k-algebra A also carries the structure of an filtered k-algebra by setting $A_{(n)} := \bigoplus_{i\leq n} A_i$ for every $n \in \mathbb{N}$. If $\varphi\colon A \to B$ is a homomorphism of graded k-algebras then it is also a homomorphism of filtered k-algebras. Hence this construction results into a functor $\mathrm{flt}\colon k\text{-}\mathbf{Grad} \to k\text{-}\mathbf{Filt}$.

2. If A is a filtered algebra and $I \subseteq A$ any two-sided ideal then A/I is a filtered k-algebra via $(A/I)_{(n)} := \pi(A_{(n)})$ for every $n \in \mathbb{N}$, where $\pi\colon A \to A/I, a \mapsto a + I$ denotes the canonical projection.

3. If $\mathfrak{g}$ is a k-Lie algebra then $\mathcal{U}(\mathfrak{g})$ carries the structure of a filtered k-algebra induced by the filtration of $T(\mathfrak{g})$, which in turn is induced by the gradation of $T(\mathfrak{g})$. Explicitely

$$\mathcal{U}(\mathfrak{g})_{(n)} = \mathrm{span}_k\{x_1 \cdots x_m \mid m \in \mathbb{N}, m \leq n, x_1, \ldots, x_m \in \mathfrak{g}\} \quad \text{for every } n \in \mathbb{N}.$$

Lemma 2.3.26. *Let A be a graded k-algebra and set $B_n := A_n/A_{n-1}$ for every $n \in \mathbb{N}$. Then for all $n, m \in \mathbb{N}$ the map*

$$B_n \times B_m \to B_{n+m}, ([a],[b]) \mapsto [ab]$$

is well-defined and bilinear.

Proof. Let $a, a' \in A_n$ and $b, b' \in A_m$ with $[a] = [a']$ and $[b] = [b']$. Then $ab, a'b' \in A_{n+m}$ and because $[a] = [a']$ and $[b] = [b']$ it follows that $a - a' \in A_{n-1}$ and $b - b' \in A_{m-1}$. Therefore

$$ab = (a' + (a - a'))(b' + (b - b')) = a'b' + \underbrace{(a - a')b'}_{\in A_{n+m-1}} + \underbrace{a'(b - b')}_{\in A_{n+m-1}} + \underbrace{(a - a')(b - b')}_{\in A_{n+m-2}}$$

and thus $[ab] = [a'b']$. □

Definition 2.3.27. Let A be a filtered k-algebra. Its *associated graded k-algebra* is the graded k-algebra consisting of the underlying vector space $\operatorname{gr}(A) := \bigoplus_{n\in\mathbb{N}} \operatorname{gr}_n(A)$ with $\operatorname{gr}_i(A) := A_{(i)}/A_{(i-1)}$ for every $i \in \mathbb{N}$ and the multiplication $\operatorname{gr}(A) \times \operatorname{gr}(A) \to \operatorname{gr}(A)$ induced by the well-defined bilinear maps

$$\operatorname{gr}_n(A) \times \operatorname{gr}_m(A) \to \operatorname{gr}_{n+m}(A), \quad ([a],[b]) \to [ab] \quad \text{for all } n,m \in \mathbb{N}.$$

together with the grading given by $\operatorname{gr}(A)_n := \operatorname{gr}_n(A)$ for every $n \in \mathbb{N}$.

Remark 2.3.28. If A and B are filtered k-algebras and $\varphi\colon A \to B$ is a homomorphism of filtered k-algebras then φ induces a k-linear map $\varphi_n\colon \operatorname{gr}(A)_n \to \operatorname{gr}(B)_n, [a] \mapsto [\varphi(a)]$ for every $n \in \mathbb{N}$, which result in a homomorphism of graded k-algebras

$$\operatorname{gr}(\varphi) := \bigoplus_{n\in\mathbb{N}} \varphi_n\colon \operatorname{gr}(A) \to \operatorname{gr}(B), \quad \sum_{n\in\mathbb{N}} a_n \mapsto \sum_{n\in\mathbb{N}} \varphi(a_n).$$

Hence gr can be seen as a functor $\operatorname{gr}\colon k\textbf{-Filt} \to k\textbf{-Grad}$.

Examples 2.3.29. 1. If A is a graded k-algebra then $\operatorname{gr}(\operatorname{flt}(A))$ is naturally isomorphic to A in the following way: The filtration on $\operatorname{flt}(A)$ is given by $A_{(n)} = \bigoplus_{i\leq n} A_i$ for every $n \in \mathbb{N}$. Hence there is for every $n \in \mathbb{N}$ an isomorphism of vector spaces

$$\varphi_n\colon A_n \to A_{(n)}/A_{(n-1)} = \operatorname{gr}(\operatorname{flt}(A))_n, \quad a \mapsto [a].$$

Combining these isomorphisms results in an isomorphism of graded k-algebras

$$\bigoplus_{n\in\mathbb{N}} \varphi_n\colon \bigoplus_{n\in\mathbb{N}} A_n \to \bigoplus_{n\in\mathbb{N}} \operatorname{gr}(\operatorname{flt}(A))_n, \quad \sum_{n\in\mathbb{N}} a_n \mapsto \sum_{n\in\mathbb{N}} [a_n].$$

We will therefore identify A with $\operatorname{gr} A$ in the above way.

2. Let $\mathfrak{g}$ be a k-Lie algebra. The canonical projection

$$\pi\colon T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}), \quad x_1 \otimes \cdots \otimes x_n \to x_1 \cdots x_n \quad \text{for every } n \in \mathbb{N} \text{ and all } x_1, \ldots, x_n \in \mathfrak{g}$$

is a homomorphism of filtered k-algebras, where the filtration of $T(\mathfrak{g})$ is induced by the gradation discussed in Examples 2.3.13. Hence it induces a homomorphism of graded k-algebras

$$\operatorname{gr}(\pi)\colon T(\mathfrak{g}) \to \operatorname{gr}(\mathcal{U}(\mathfrak{g})),$$

where $T(\mathfrak{g})$ is identified with $\operatorname{gr}(T(\mathfrak{g}))$ as above. This homomorphism maps an element $x_1 \otimes \cdots \otimes x_n$ with $x_1, \ldots, x_n \in \mathfrak{g}$ to the residue class $[x_1 \cdots x_n] \in \operatorname{gr}(\mathcal{U}(\mathfrak{g}))_n$.

Proposition 2.3.30. *Let A be a filtered k-algebra. If $\operatorname{gr}(A)$ is an integral domain then so is A.*

Proof. Suppose A is no integral domain. Then there exist $a, b \in A$ with $a \neq 0$ and $b \neq 0$ but $ab = 0$. Then there exists a minimal $n \in \mathbb{N}$ with $a \in A_{(n)}$ and a minimal $b \in \mathbb{N}$ with $b \in A_{(m)}$. By the minimality of n and m it follows that $[a] \in \operatorname{gr}(A)_n$ and $[b] \in \operatorname{gr}(A)_m$ are nonzero residue classes with $[a] \cdot [b] = [ab] = 0$. Hence $\operatorname{gr}(A)$ is no integral domain. □

Remark 2.3.31. The converse of Proposition 2.3.30 is not true, i.e. if A is a filtered k-algebra which is an integral domain, then $\mathrm{gr}(A)$ is not necessarily an integral domain.

To see this let A be any k-algebra with $k \subsetneq A$ and with filtration

$$A_{(0)} \coloneqq k \quad \text{and} \quad A_{(n)} \coloneqq A \quad \text{for every } n \geq 1.$$

Then $\mathrm{gr}(A)_0 = k$, $\mathrm{gr}(A)_1 = A/k \neq 0$ and $\mathrm{gr}(A)_n = 0$ for every $n \geq 2$, hence $\mathrm{gr}(A)_1 \,\mathrm{gr}(A)_1 = 0$. So $\mathrm{gr}(A)$ is no integral domain, even if A is.

2.3.2.3. The Poincaré–Birkhoff–Witt theorem (concrete version)

For this subsection we fix some k-Lie algebra $\mathfrak{g}$ with basis $(x_i)_{i\in I}$ where $(I, \leq)$ is a totally ordered index set. Before stating and proving the *Poincaré–Birkhoff–Witt theorem* (PBW) we fix some notation which we will only use in this subsection.

Definition 2.3.32. Let $\mathcal{I}_n \coloneqq \{(i_1, \dots, i_n) \mid i_1, \dots, i_n \in I,\ i_1 \leq \dots \leq i_n\}$ for every $n \in \mathbb{N}$ and set $\mathcal{I} \coloneqq \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$. For every $\alpha = (i_1, \dots, i_n) \in I^n$ with $n \in \mathbb{N}$ let $x_\alpha \coloneqq x_{i_1} \cdots x_{i_n} \in \mathcal{U}(\mathfrak{g})$.

Remark 2.3.33. Notice that $\mathcal{I}_0$ contains the empty tupel.

Theorem 2.3.34 (PBW (concrete version))**.** *The familiy $(x_\alpha \mid \alpha \in \mathcal{I})$ is a k-basis of $\mathcal{U}(\mathfrak{g})$.*

Remark 2.3.35. The basis $(x_\alpha \mid \alpha \in \mathcal{I})$ can also be written as

$$\left(x_{i_1}^{p_1} \cdots x_{i_n}^{p_n} \mid n \in \mathbb{N},\ i_1, \dots, i_n \in I,\ i_1 < \dots < i_n,\ p_1, \dots, p_n \geq 1\right).$$

Example 2.3.36. If $\mathfrak{g}$ is a finite dimensional k-Lie algebra with basis $x_1, \dots, x_n$ then $\mathcal{U}(\mathfrak{g})$ has a basis given by $(x_1^{p_1} \cdots x_n^{p_n} \mid p_1, \dots, p_n \in \mathbb{N})$. In particular a basis of $\mathcal{U}(\mathfrak{sl}_2(k))$ is given by $(e^\ell h^m f^n \mid \ell, m, n \in \mathbb{N})$.

Lemma 2.3.37. *The collection $(x_\alpha \mid \alpha \in \mathcal{I})$ generates $\mathcal{U}(\mathfrak{g})$ as a vector space.*

Proof. To show that $(x_\alpha \mid \alpha \in \mathcal{I})$ generates $\mathcal{U}(\mathfrak{g})$ as a vector space it sufficies to show that $\mathcal{B}_n \coloneqq (x_\alpha \mid m \leq n, \alpha \in \mathcal{I}_m)$ generates $\mathcal{U}(\mathfrak{g})_{(n)}$ as a vector space, which will be shown by induction over $n \in \mathbb{N}$: For $n = 0$ it holds because $\mathcal{U}(\mathfrak{g}) = k$ is one-dimensional and thus spanned by $x_{(\,)}$, the monomial corresponding to the empty tupel.

Suppose that the statement holds for some $n \in \mathbb{N}$. Then $\mathcal{U}(\mathfrak{g})_{(n+1)}$ is generated as a vector space as a by the monomials $(x_{(i_1,\dots,i_n)} \mid m \leq n+1,\ i_1, \dots, i_m \in I)$. Therefore it sufficies to express these monomials in terms of $\mathcal{B}_{n+1}$. By induction hypothesis is it enough to check this for the monomials $(x_{(i_1,\dots,i_{n+1})} \mid i_1, \dots, i_{n+1} \in I)$. For this let $\alpha = (i_1, \dots, i_{n+1})$ be some fixed multiindex with $i_1, \dots, i_{n+1} \in I$.

For any two $x, y \in \mathfrak{g}$ one has $xy = yx + [x, y]$ with $[x, y] \in \mathcal{U}(\mathfrak{g})_{(1)}$. Hence there exists for any permutation $\sigma \in S_{n+1}$ a linear combination $R_\sigma \in \mathcal{U}(\mathfrak{g})_{(1)}$ with

$$x_\alpha = x_{i_1} \cdots x_{i_{n+1}} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n+1)}} + R_\sigma = x_{(i_{\sigma(1)},\dots,x_{\sigma(n+1)})} + R_\sigma.$$

Let $\sigma \in S_{n+1}$ be a permutation with $i_{\sigma(1)} \leq \dots \leq i_{\sigma(n+1)}$. By induction hypothesis $R_\sigma \in \mathcal{U}(\mathfrak{g})_{(n)}$ can be expressed as a linear combination of the monomials $\mathcal{B}_n$. Hence $x_\alpha = x_{(i_{\sigma(1)},\dots,i_{\sigma(n+1)})} + R_\sigma$ can be expressed as a linear combination of the monomials $\mathcal{B}_{n+1}$ because $x_{(i_{\sigma(1)},\dots,i_{\sigma(n+1)})}$ is one of them. $\square$

Proof of PBW (concrete version). By Lemma 2.3.37 the collection $(x_\alpha \mid \alpha \in \mathcal{I})$ generates $\mathcal{U}(\mathfrak{g})$ as a vector space, so all that's left to show is that it is linearly independent.

Let $V := k[Z_i \mid i \in I]$ and for every $n \in \mathbb{N}$ let $V_{(n)}$ be the polynomials of degree $\leq n$. For every $n \in \mathbb{N}$ and $\alpha = (i_1, \dots, i_n) \in I^n$ write $Z_\alpha := Z_{i_1} \cdots Z_{i_n}$. If $i \in I$ and $\alpha = (i_1, \dots, i_n) \in I^n$ then write $i \leq \alpha$ if $i \leq i_j$ for every $j = 1, \dots, n$. Also set $i \cdot \alpha = (i, i_1, \dots, i_n) \in I^{n+1}$.

To show that $(x_\alpha \mid \alpha \in \mathcal{I})$ is linearly independent V will be given the structure of a representation of $\mathfrak{g}$ such that

$$x_i.Z_\alpha = Z_{i\cdot\alpha} \quad \text{for every } i \in I \text{ and } \alpha \in \mathcal{I} \text{ with } i \leq \alpha.$$

Then for the corresponding $\mathcal{U}(\mathfrak{g})$-module structure on V it follows that

$$x_\alpha \cdot 1 = x_\alpha \cdot Z_{(\,)} = Z_\alpha \quad \text{for every } \alpha \in \mathcal{I}$$

where $1 \in V = k[Z_i \mid i \in I]$ and $(\,)$ denotes the empty tupel. Because $(Z_\alpha \mid \alpha \in \mathcal{I})$ is linearly independent it then follows that $(x_\alpha \mid \alpha \in \mathcal{I})$ is linearly independent. The existence of such an action follows from the following:

Claim. *There exists a unique sequence $(\varphi_n)_{n\in\mathbb{N}}$ of bilinear maps*

$$\varphi_n \colon \mathfrak{g} \times V_{(n)} \to V_{(n+1)}, \quad (x, p) \mapsto x.p$$

satisfying the following conditions:

1. *The restriction of φ_{n+1} to $\mathfrak{g} \times V_{(n)}$ coincides with φ_n for every $n \in \mathbb{N}$*
2. *$x_i.Z_\alpha = Z_{i\cdot\alpha}$ for every $i \in I$ and $\alpha \in \mathcal{I}$ with $i \leq \alpha$.*
3. *$x_i.x_j.Z_\alpha - x_j.x_i.Z_\alpha = [x_i, x_j].Z_\alpha$ for all $i, j \in I$ and every $\alpha \in \mathcal{I}$.*
4. *$x_i.Z_\alpha - Z_{i\cdot\alpha} \in V_{(n)}$ for every $n \in \mathbb{N}$, $i \in I$ and $\alpha \in \mathcal{I}_n$.*

*(Condition 1 actually follows from the other conditions by the uniqueness of the sequence $(\varphi_n)_{n\in\mathbb{N}}$. See [**Humphreys**] for more details.)*

Proof. Notice thet the notation $x.p$ with $x \in \mathfrak{g}$ and $p \in V$ is unambiguous by condition 1. The maps φ_n will be defined by induction over n:

As $V_{(0)}$ is one-dimensional and spanned by $1 = Z_{(\,)}$ it follows from condition 2 that $x_i.1 = x_i \cdot Z_{(\,)} = Z_{i\cdot(\,)} = Z_i$ for every $i \in I$. This defines φ_0 uniquely. Conditions 2 and 4 hold by construction and the conditions 1 and 3 do not affect φ_0.

Let $n \in \mathbb{N}$ and suppose φ_m is constructed for every $m \leq n$. By condition 1 all that is left to define is $x_i.Z_\alpha$ for $\alpha \in \mathcal{I}_{n+1}$. If $i \leq \alpha$ then $x_i.Z_\alpha = Z_{i\cdot\alpha}$ by condition 2.

If $i > \alpha$ then there exists $\beta \in \mathcal{I}_n$ and $j \in I$ with $\alpha = j \cdot \beta$ such that $i > j$ and $j \leq \beta$. If condition 3 was to hold for φ_{n+1} it follows that

$$x_i.Z_\alpha = x_i.x_j.Z_\beta = x_j.x_i.Z_\beta + [x_i, x_j].Z_\beta. \tag{5}$$

Because $\beta \in \mathcal{I}_n$ the term $[x_i, x_j].Z_\beta$ in the above sum is already defined. Because $x_i.Z_\beta \equiv Z_{i\cdot\beta} \pmod{V_{(n)}}$ there exists some $Y \in V_{(n)}$ with $x_i.Z_\beta = Z_{i\cdot\beta} + Y$. Let

$\gamma \in \mathcal{I}_{n+1}$ be defined by taking β and inserting i at the right position. Then $Z_\gamma = Z_{i\cdot\beta}$. Because $j < i$ and $j \le \beta$ it follows that $j \le \gamma$ and therefore

$$x_j.Z_{i\cdot\beta} = x_j.Z_\gamma = Z_{i\cdot\gamma} = Z_{i\cdot(j\cdot\beta)} = Z_{i\cdot\alpha}$$

Because every summand in

$$\begin{aligned} x_i.Z_\alpha = x_j.x_i.Z_\beta + [x_i, x_j].Z_\beta &= x_j.(Z_{i\cdot\beta} + Y) + [x_i, x_j].Z_\beta \\ &= Z_{i\cdot\alpha} + x_j.Y + [x_i, x_j].Z_\beta \end{aligned} \tag{6}$$

is defined it follows that φ_{n+1} is uniquely defined.

Conditions 1 and 2 hold for φ_{n+1} by construction. Condition 4 holds for $i \le \alpha$ by condition 2 and for $i > \alpha$ by (6) because $x_j.Y \in V_{(n+1)}$ and $[x_i, x_j].Z_\beta \in V_{(n+1)}$.

It remains to check Condition 3 for φ_{n+1}, i.e. when $i, j \in I$ and $\alpha \in \mathcal{I}_n$. For $i = j$ this follows from the Lie bracket being alternating. Suppose that $i \ne j$. By the Lie bracket is antisymmetric it can be w.l.o.g. assumed that $i < j$.

If $i \le \alpha$ then $x_j.x_i.Z_\alpha$ is defined above by using (5) (where β has to be replaced by α and i and j have to be switched), hence condition 2 holds in this case by construction. Notice that if $j \le \alpha$ then also $i \le \alpha$ because $i < j$.

Hence the only case left is $i \not\le \alpha$. By the above it then follows that also $j \not\le \alpha$. As this cannot happen for $n = 0$ it can be w.l.o.g. assumed that $n \ge 1$. Let $k \in I$ and $\beta \in \mathcal{I}_{n-1}$ with $\alpha = k \cdot \beta$. Because condition 3 holds for φ_n it follows that

$$\begin{aligned} x_i.x_j.Z_\alpha = x_i.x_j.x_k.Z_\beta &= x_i.(x_k.x_j.Z_\beta + [x_j, x_k].Z_\beta) \\ &= x_i.x_k.x_j.Z_\beta + x_i.[x_j, x_k].Z_\beta \end{aligned}$$

Because $k < j$ and $k \le \beta$ it follows from the previous discussed cases that

$$x_i.x_k.(x_j.Z_\beta) = x_k.x_i.(x_j.Z_\beta) + [x_i, x_k].(x_j.Z_\beta).$$

Combining the above results in the equality

$$x_i.x_j.Z_\alpha = x_k.x_i.x_j.Z_\beta + [x_i, x_k].x_j.Z_\beta + x_i.[x_j, x_k].Z_\beta$$

By switching i and j in the above calculations it also follows that

$$x_j.x_i.Z_\alpha = x_k.x_j.x_i.Z_\beta + [x_j, x_k].x_i.Z_\beta + x_j.[x_i, x_k].Z_\beta$$

By using that condition 3 holds for φ_n it follows from these two equalities and the Jacobi identity that

$$\begin{aligned} x_i.x_j.Z_\alpha - x_j.x_i.Z_\alpha &= x_k.x_i.x_j.Z_\beta + [x_i, x_k].x_j.Z_\beta + x_i.[x_j, x_k].Z_\beta \\ &\quad - x_k.x_j.x_i.Z_\beta - [x_j, x_k].x_i.Z_\beta - x_j.[x_i, x_k].Z_\beta \\ &= x_k.[x_i, x_j].Z_\beta + [[x_i, x_k], x_j].Z_\beta + [x_i, [x_j, x_k]].Z_\beta \\ &= x_k.[x_i, x_j].Z_\beta - [x_k, [x_i, x_j]].Z_\beta = [x_i, x_j].x_k.Z_\beta = [x_i, x_j].Z_\alpha. \square \end{aligned}$$

This finishes the proof. □

Corollary 2.3.38. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}, \mathfrak{n} \subseteq \mathfrak{g}$ Lie subalgebras with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ as vector spaces. Then the map*

$$\mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{g}), \quad x \otimes y \mapsto xy$$

is a isomorphism of vector spaces.

Proof. Let $(x_i)_{i\in I}$ is a basis of $\mathfrak{h}$ and $(x_j)_{j\in J}$ a basis of $\mathfrak{n}$. Then $(x_k)_{k\in K}$ for the index set $K := I \cup J$ is a basis of $\mathfrak{g}$. Then the statement follows directly from the concrete PBW theorem. □

2.3.2.4. The Poincaré–Birkhoff–Witt theorem (abstract version)

Theorem 2.3.39 (PBW (abstract version))**.** *Let $\mathfrak{g}$ be a Lie algebra over k and denote by π the canonical projection*

$$\pi\colon T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n \quad \text{for all } x_1, \ldots, x_n \in \mathfrak{g}.$$

Then the two homomorphisms of graded k-algebras $\operatorname{gr}(\pi)\colon T(\mathfrak{g}) \to \operatorname{gr}(\mathcal{U}(\mathfrak{g}))$ and

$$\pi'\colon T(\mathfrak{g}) \to S(\mathfrak{g}), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n \quad \text{for all } x_1, \ldots, x_n \in \mathfrak{g}$$

have the same kernel and thus induce an isomorphism of graded algebras

$$\varphi\colon S(\mathfrak{g}) \to \operatorname{gr}(\mathcal{U}(\mathfrak{g})), \quad x_1 \cdots x_n \mapsto [x_1 \cdots x_n] \quad \text{for all } x_1, \ldots, x_n \in \mathfrak{g}. \tag{7}$$

Remark 2.3.40. Notice that the two multiplications in (7) live in different k-algebras.

Proposition 2.3.41. *The concrete version and the abstract versions of the PBW-theorem are equivalent.*

Proof. For $x, y \in \mathfrak{g}$ it follows from the definition of $\operatorname{gr}(\pi)$ that

$$\operatorname{gr}(\pi)(x \otimes y - y \otimes x) = [xy - yx] \in \operatorname{gr}(\mathcal{U}(\mathfrak{g}))_2,$$

with representative $xy - yx \in \mathcal{U}(\mathfrak{g})_{(2)}$. By the definition of $\mathcal{U}(\mathfrak{g})$ it follows that already $xy - xy = [x, y] \in \mathcal{U}(\mathfrak{g})_{(1)}$. Hence it follows for the residue class of $xy - yx$ in $\operatorname{gr}(\mathcal{U}(\mathfrak{g}))_2 = \operatorname{gr}(\mathcal{U}(\mathfrak{g}))_{(2)} / \operatorname{gr}(\mathcal{U}(\mathfrak{g}))_{(1)}$ that $[xy - yx] = [[x, y]] = [0] = 0$. Hence $\operatorname{gr}(\pi)(x \otimes y - y \otimes x) = 0$.

As the kernel of π' is generated by the element $x \otimes y - y \otimes x$ with $x, y \in \mathfrak{g}$ it follows that π' factorizes through a homomorphism of graded k-algebras φ as in Theorem 2.3.39.

(concrete ⇒ abstract) The algebra $\mathcal{U}(\mathfrak{g})$ has a basis $(x_\alpha \mid \alpha \in \mathcal{I})$ It follows that $\operatorname{gr}(\mathcal{U}(\mathfrak{g}))_n$ has a basis given by the residue classes $([x_\alpha] \mid \alpha \in \mathcal{I}_n)$. The linear subspace $S(\mathfrak{g})_n$ has a basis $(x_1 \cdots x_n \mid (i_1, \ldots, i_n) \in \mathcal{I}_n)$ which is mapped by φ_n to the above basis of $\operatorname{gr}(\mathcal{U}(\mathfrak{g}))_n$. Hence φ_n is an isomorphism for every $n \in \mathbb{N}$, which is why φ is an isomorphism.

(abstract ⇒ concrete) As $(x_\alpha \mid \alpha \in \mathcal{I})$ generates $\mathcal{U}(\mathfrak{g})$ as a vector spaces by Lemma 2.3.37 it suffices to show that this collection is linearly independent. Suppose otherwise. Then there exists some minimal $n \in \mathbb{N}$ such that $(x_\alpha \mid m \leq n, \alpha \in \mathcal{I}_m)$ is linearly dependent. Hence there exists a non-trivial linear combination

$$0 = \sum_{m=0}^{n} \sum_{\alpha \in \mathcal{I}_m} \lambda_\alpha x_\alpha \quad \text{where } \lambda_\alpha = 0 \text{ for all but finitely many } \alpha \in \bigcup_{m=0}^{n} \mathcal{I}_m.$$

From this it follows that

$$0 = \sum_{m=0}^{n} \sum_{\alpha \in \mathcal{I}_m} \lambda_\alpha x_\alpha \equiv \sum_{\alpha \in \mathcal{I}_n} \lambda_\alpha x_\alpha \mod \mathcal{U}(g)_{(n-1)}$$

and hence that the equality $\sum_{\alpha \in \mathcal{I}_n} \lambda_\alpha [x_\alpha] = 0$ holds in $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))_n = \mathcal{U}(\mathfrak{g})_{(n)}/\mathcal{U}(\mathfrak{g})_{(n-1)}$. By the minimality of n it follows that this is a non-trivial linear combination in $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))_n$, so $([x_\alpha] \mid \alpha \in \mathcal{I}_n)$ in $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))_n$ is linearly dependent.

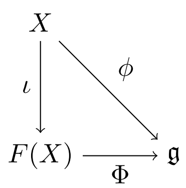
By assumption φ is a homomorphism of graded k-algebras and therefore φ_n is an isomorphism of vector spaces. As the basis $(x_{i_1} \cdots x_{i_n} \mid (i_1, \ldots, i_n) \in \mathcal{I}_n)$ of $S(\mathfrak{g})_n$ is mapped by φ_n bijectively to $([x_\alpha] \mid \alpha \in \mathcal{I}_n)$ it follows that this is a basis of $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))_n$, contradicting the linearly dependency. □

Corollary 2.3.42. *The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is an integral domain.*

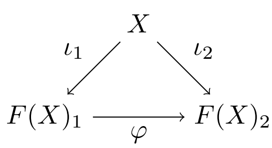
Proof. Because $\mathrm{gr}(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{g})$ is in integral domain the statement follows from Proposition 2.3.30. □

2.3.3. Free Lie algebras

Definition 2.3.43. Let X be a set. A *free k-Lie algebra on X* is a Lie algebra $F(X)$ together with a map $\iota\colon X \to F(X)$ such that for every Lie algebra $\mathfrak{g}$ and map $\phi\colon X \to \mathfrak{g}$ there exists a unique homomorphism of Lie algebras $\Phi\colon F(X) \to \mathfrak{g}$ with $\phi = \Phi \circ \iota$, i.e. making the following diagram commute:



Remark 2.3.44. As usual with free objects it follows that any two free Lie algebras over a set X are unique up to unique isomorphism, i.e. if $F(X)_1$ with $\iota_1\colon X \to F(X)_1$ and $F(X)_2$ with $\iota_2\colon X \to F(X)_2$ are two free Lie algebras over X then there exists a unique isomorphism of Lie algebras $\varphi\colon F(X)_1 \to F(X)_2$ making the following diagram commute:



We will therefore always talk about *the* free k-Lie algebra over X.

Lemma 2.3.45. *Let X be a set. Then there exists a free Lie algebra over X.*

Proof. Let $A(X)$ be the free (unitary and associative) k-algebra over X (which can be constructed as $T(kX)$, i.e. the tensor algebra over the free vector space kX with basis X). Let $F(X)$ be the Lie subalgebra of $A(X)$ generated by X, i.e.

$$F(X) = \bigcap \{\mathfrak{g} \mid \mathfrak{g} \subseteq A(X) \text{ is a Lie subalgebra with } X \subseteq \mathfrak{g}\}.$$

Let $\mathfrak{g}$ be a k-Lie algebra and $\phi\colon X \to \mathfrak{g}$ a map. By the universal property of the free k-algebra the map ϕ induces a homomorphism of k-algebras $\theta\colon A(X) \to \mathcal{U}(\mathfrak{g})$ making the following diagram commute, where the vertical maps are the canonical inclusions:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathfrak{g} \\ \downarrow & & \downarrow \\ F(X) & \xrightarrow[\theta]{} & \mathcal{U}(\mathfrak{g}) \end{array}$$

As $\theta(X) = \phi(X) \subseteq \mathfrak{g}$ it follows that $X \subseteq \theta^{-1}(\mathfrak{g})$. Because $\theta^{-1}(\mathfrak{g})$ is a Lie subalgebra of $F(X)$ containing X it follows that $\theta^{-1}(\mathfrak{g}) = F(X)$ and therefore $\theta(F(X)) \subseteq \mathfrak{g}$. Hence θ restricts to a map $\Phi\colon F(X) \to \mathfrak{g}$. Because θ is a homomorphism of k-algebras it is in particular a homomorphism of Lie algebras and therefore the same goes for Φ. This shows the existence.

For the uniqueness notice that $F(X)$ is by definition generated by X and hence any homomorphism of Lie algebras $\Psi\colon F(X) \to \mathfrak{g}$ is uniquely determinad by the restriction $\Psi|_X$. □

Remark 2.3.46. The universal enveloping algebra is used in the proof of Lemma 2.3.45 to ensure that any Lie algebra can be embedded into a k-algebra as a Lie subalgebra.

Remark 2.3.47. Using the concept of free Lie algebras one can define Lie algebras by giving a set of generators X and a set of relations $R \subseteq F(X)$. As an example the Lie algebra $\mathfrak{sl}(k)$ can be defined by the generators $R \coloneqq \{e, h, f\}$ with e, h, f being pairwise different and the relations $R = \{[h, e] - 2e, [h, f] + 2f, [e, f] - h\}$, which can also be written as $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$ as usual.

More generally $\mathfrak{sl}_{n+1}(k)$ for $n \geq 1$ can be defined as the Lie algebra generated by the $3n$ elements $\{e_i, f_i, h_i \mid i = 1, \ldots, n\}$ together with the relations

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, e_j] &= a_{ij} e_j, \\ [h_i, f_j] &= -a_{ij} f_j, \\ [e_i, f_j] &= \delta_{ij} h_i, \end{aligned}$$

for all $i, j = 1, \ldots, n$ and

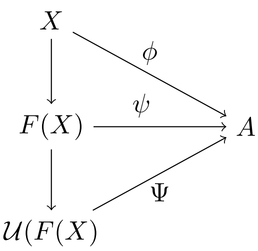
$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \quad \text{and} \quad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } 1 \leq i \neq j \leq n,$$

where the numbers a_{ij} are for all $i, j = 1, \ldots, n$ defined as

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3.48. *Let X be any set and $F(X)$ the free Lie algebra over X. Then $\mathcal{U}(F(X))$ is the free k-algebra over X, where the canonical inclusion $X \hookrightarrow \mathcal{U}(F(X))$ is given by composition of the canonical inclusions $X \hookrightarrow F(X)$ and $F(X) \hookrightarrow \mathcal{U}(F(X))$.*

Proof. It will be shown that $\mathcal{U}(F(X))$ together with the canonical inclusion $X \hookrightarrow \mathcal{U}(F(X))$ satisfies universal property of the free k-algebras. Let A be a k-algebra and $\phi\colon X \to A$ a map. Then ϕ induces a unique homomorphism of Lie algebras $\psi\colon F(X) \to A$ by the universal property of the free Lie algebra. Then ψ induces a unique homomorphism of k-algebras $\Psi\colon \mathcal{U}(F(X)) \to A$ by the universal property of the universal enveloping algebra. Hence the following diagram commutes, where the vertical maps denote the canonical inclusions.



Leaving out the middle part of the diagram shows the existence. The uniqueness of Ψ follows from the uniqueness of ψ. □

2.3.4. Casimir elements

For this subsection we additionaly assume that $\mathfrak{g}$ is finite-dimensional. We also fix some bilinear form $\beta\colon \mathfrak{g} \times \mathfrak{g} \to k$ which is associative and non-degenerate.

Definition 2.3.49. Let $\varphi_1\colon \mathfrak{g} \otimes \mathfrak{g}^* \to \mathrm{End}_k(\mathfrak{g})$ and $\varphi_2\colon \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphisms of vector spaces defined by

$$\varphi_1(x \otimes \phi)(y) = \phi(y)x \quad \text{and} \quad \varphi_2(x) = \beta(x, \cdot) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } \phi \in \mathfrak{g}^*.$$

Then the image of 1 under the map

$$k \xrightarrow{\lambda \mapsto \lambda\, \mathrm{id}_{\mathfrak{g}}} \mathrm{End}_k(\mathfrak{g}) \xrightarrow{\varphi_1^{-1}} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\mathrm{id}_{\mathfrak{g}} \otimes \varphi_2^{-1}} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{x \otimes y \mapsto xy} \mathcal{U}(\mathfrak{g}) \tag{8}$$

is called the *Casimir element of* β and denoted by C_β.

Lemma 2.3.50. *The Casimir element C_β in central in $\mathcal{U}(\mathfrak{g})$, i.e.*

$$xC_\beta = C_\beta x \quad \textit{for every } x \in \mathcal{U}(g).$$

Proof. Let φ_1 and φ_2 as in Definition 2.3.49.

Because $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$ as a k-algebra it sufficies to show C_β commutes with every $x \in \mathfrak{g}$. Hence it is to show that

$$[x, C_\beta] = 0 \quad \text{for every } x \in \mathfrak{g},$$

where $[\cdot,\cdot]$ denotes the Lie bracket in $\mathcal{U}(\mathfrak{g})$. To see this notice that in (8) every map is a homomorphism of representations of $\mathfrak{g}$, where $\mathfrak{g}$ acts trivially on k, i.e. $x.\lambda = 0$ for every $x \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}$.

That the first map $k \to \operatorname{End}_k(\mathfrak{g})$ is a homomorphism of representations follows from the fact that $\mathfrak{g}$ acts trivially on k and also trivially on the one-dimensional subspace $k \operatorname{id}_{\mathfrak{g}} \subseteq \operatorname{End}_k(\mathfrak{g})$.

That φ_1 is an isomorphism of representations is known from Propositon 1.2.14.

That the third map $\mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is a homomorphism of representations follows from Proposition 1.2.14, because the identity $\operatorname{id}_{\mathfrak{g}}$ is a homomorphism of representations and and the isomorphism φ_2 is one by the associativity of β, as seen in Lemma 2.2.3.

That the fourth map $\psi\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathcal{U}(\mathfrak{g}), x \otimes y \mapsto xy$ is a homomorphism of representations follows from direct calculation, because for all $x, y, z \in \mathfrak{g}$

$$\begin{aligned}
\psi(x.(y \otimes z)) &= \psi((x.y) \otimes z + y \otimes (x.z)) = (x.y)z + y(x.z) \\
&= [x, y]z + y[x, z] = xyz - yxz + yxz - yzx = xyz - yzx \\
&= [x, yz] = x.(yz) = x.\psi(y \otimes z).
\end{aligned}$$

Because every map in (8) is a homomorphism of representations it follows that their composition $\phi\colon k \to \mathcal{U}(\mathfrak{g})$ is also a homomorphism of representations. Definition 2.3.49 is then equivalent to $\phi(1) = C_\beta$. Because $\mathfrak{g}$ acts trivially on k and ϕ is a homomorphism of representations it follows that $\mathfrak{g}$ also acts trivially on the span of C_β. In particular

$$0 = x.C_\beta = [x, C_\beta] \quad \text{for every } x \in \mathfrak{g}.$$

□

Corollary 2.3.51. *Let V be a representation of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g}) \times V \to V, (x, v) \mapsto x \cdot v$ the corresponding $\mathcal{U}(\mathfrak{g})$-module structure on V. Then the map*

$$C_\beta^V : V \to V, \quad v \mapsto C_\beta \cdot v = \sum_{i=1}^n x_i.x^i.v$$

is an endomorphism of representations of $\mathfrak{g}$ (equivalently an endomorphism of $\mathcal{U}(\mathfrak{g})$-modules).

Proof. Because $C_\beta \in Z(\mathcal{U}(\mathfrak{g}))$ it follows that for every $x \in \mathcal{U}(\mathfrak{g})$ and $v \in V$

$$x \cdot C_\beta^V(v) = x \cdot C_\beta \cdot v = C_\beta \cdot x \cdot v = C_\beta^V(x \cdot v).$$

□

Lemma 2.3.52 (Casimir in coordinates)**.** *Let $x_1, \dots, x_n$ be a basis of $\mathfrak{g}$ and $x^1, \dots, x^n$ the dual basis of $\mathfrak{g}$ with respect to β, i.e. $\beta(x_i, x^j) = \delta_{ij}$ for all $i, j = 1, \dots, n$. Then*

$$C_\beta = \sum_{i=1}^{n} x_i x^i.$$

Proof. Let φ_1 and φ_2 as in Definition 2.3.49. In (8) 1 is mapped to $\mathrm{id}_{\mathfrak{g}}$, which is then mapped to $\sum_{i=1}^n x_i \otimes x_i^*$, where $x_1^*, \dots, x_n^*$ denotes the dual basis of $\mathfrak{g}^*$. As $\varphi_2(x^i) = x_i^*$ it follows that $\sum_{i=1}^n x_i \otimes x_i^*$ is then mapped to $\sum_{i=1}^n x_i \otimes x^i$, which is then further mapped to the element $\sum_{i=1}^n x_i x^i$ in $\mathcal{U}(\mathfrak{g})$. □

Remark 2.3.53. Using Lemma 2.3.52 it can be shown that C_β is central in $\mathcal{U}(\mathfrak{g})$ using coordinates: Let $x \in \mathfrak{g}$ and $a_{ij}, b_{ij} \in k$ such that $[x, x_i] = \sum_{j=1}^n a_{ij} x_j$ and $[x, x^i] = \sum_{j=1}^n b_{ij} x^j$ for all $i = 1, \dots, n$. Then for all $i, j = 1, \dots, n$

$$\begin{aligned} a_{ij} &= \sum_{k=1}^{n} a_{ik} \beta(x_k, x^j) = \beta\left(\sum_{k=1}^{n} a_{ik} x_k, x^j\right) = \beta([x, x_i], x^j) = -\beta([x_i, x], x^j) \\ &= -\beta(x_i, [x, x^j]) = -\beta\left(x_i, \sum_{k=1}^{n} b_{jk} x^k\right) = -\sum_{k=1}^{n} b_{jk} \beta(x_i, x^k) = -b_{ji}. \end{aligned}$$

It follows that

$$\begin{aligned} xC_\beta - C_\beta x &= \sum_{i=1}^{n} x x_i x^i - \sum_{i=1}^{n} x_i x^i x = \sum_{i=1}^{n} [x, x_i] x^i - \sum_{i=1}^{n} x_i [x^i, x] \\ &= \sum_{i,j=1}^{n} a_{ij} x_j x^i + \sum_{i,j=1}^{n} b_{ij} x_i x^j = \sum_{i,j=1}^{n} a_{ij} x_j x^i - \sum_{i,j=1}^{n} a_{ij} x_j x^i = 0. \end{aligned}$$

3. Semisimple Lie Algebras

3.1. Definition and basic properties

Definition 3.1.1. A Lie algebra $\mathfrak{g}$ is called *semisimple* if it is the sum of finitely many simple ideals, i.e. if there exists ideals $I_1, \dots, I_n \trianglelefteq \mathfrak{g}$ which are simple (as Lie algebras) such that $\mathfrak{g} = I_1 \oplus \cdots \oplus I_n$.

Lemma 3.1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$.*

Proof. Let $I_1, \dots, I_n \trianglelefteq \mathfrak{g}$ be simple ideals such that $\mathfrak{g} = I_1 \oplus \cdots \oplus I_n$. Then

$$[\mathfrak{g}, \mathfrak{g}] = [I_1, I_1] \oplus \cdots \oplus [I_n, I_n] = I_1 \oplus \cdots \oplus I_n = \mathfrak{g}$$

as well as

$$Z(\mathfrak{g}) = Z(I_1) \oplus \cdots \oplus Z(I_n) = 0 \oplus \cdots \oplus 0 = 0.$$ □

Corollary 3.1.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$. Then $\rho(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$, i.e. $\operatorname{tr}(\rho(x)) = 0$ for every $x \in \mathfrak{g}$.*

Proof. Notice that $\rho(\mathfrak{g}) = \rho([\mathfrak{g}, \mathfrak{g}]) = [\rho(\mathfrak{g}), \rho(\mathfrak{g})] \subseteq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$. □

Corollary 3.1.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $I_1, \dots, I_n \trianglelefteq \mathfrak{g}$ simple ideals with $\mathfrak{g} = I_1 \oplus \cdots \oplus I_n$. Then the ideals $I_1, \dots, I_n$ are unique up to reordering.*

Proof. Let $J_1, \dots, J_m \trianglelefteq \mathfrak{g}$ be simple ideals with $\mathfrak{g} = J_1 \oplus \cdots \oplus J_m$. Then

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [I_1 \oplus \cdots \oplus I_n, J_1 \oplus \cdots \oplus J_m] = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} [I_i, J_j].$$

For every $i = 1, \dots, n$ and $j = 1, \dots, m$ the ideals I_i and J_j are simple, which is why

$$[I_i, J_j] = \begin{cases} I_i & \text{if } I_i = J_j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for every $i = 1, \dots, n$ there exists some $1 \leq j \leq m$ with $I_i = J_j$, and similarly that for every $j = 1, \dots, m$ there exists some $1 \leq i \leq n$ with $J_j = I_i$. Hence $n = m$ and the statement follows. □

Remark 3.1.5. If $\mathfrak{g}$ is a semisimple Lie algebra then we will often just talk about *the* decomposition $\mathfrak{g} = I_1 \oplus \cdots \oplus I_n$ into simple ideals, as this decomposition is unique up to reordering of the summands by Corollary 3.1.4.

Theorem 3.1.6. *For a finite dimensional Lie algebra $\mathfrak{g}$ the following are equivalent:*

1. $\mathfrak{g} \cong \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ *for some* $r \in \mathbb{N}$ *and simple Lie algebras* $\mathfrak{g}_i$*,* $i = 1, \dots, r$*.*

2. $\mathfrak{g} = I_1 \oplus \cdots \oplus I_s$ *for some* $s \in \mathbb{N}$ *and simple Ideals* $I_1, \dots, I_s \trianglelefteq \mathfrak{g}$*.*

3. *The Killing form* κ *of* $\mathfrak{g}$ *is non-degenerate (which is equivalent to* $\operatorname{rad} \kappa = 0$*).*

4. $\mathfrak{g}$ *has no nonzero solvable ideals (which is equivalent to* $\operatorname{rad} \mathfrak{g} = 0$*).*

5. $\mathfrak{g}$ *has no nonzero abelian ideals.*

Proof. ($4 \Rightarrow 3$) This directly follows from the fact that $\operatorname{rad} \kappa$ is a solvable ideal in $\mathfrak{g}$.

($3 \Rightarrow 2$) The implication can be shown by induction over $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 0$ then $\mathfrak{g} = 0$ is the empty sum over zero simple ideals in $\mathfrak{g}$. Suppose that $\dim \mathfrak{g} \geq 1$ and the implication holds for all smaller dimensions. If $\mathfrak{g}$ is simple then there is nothing left to show. Otherwise $\mathfrak{g}$ contains a non-trivial ideal $I \trianglelefteq \mathfrak{g}$, i.e. $I \neq 0$ and $I \neq \mathfrak{g}$. Then

$$I^{\perp} := \{y \in \mathfrak{g} \mid \kappa(x, y) = 0 \text{ for every } x \in I\}$$

is an ideal in $\mathfrak{g}$ and because κ is non-degenerate it follows that $\dim \mathfrak{g} = \dim I + \dim I^{\perp}$. Because $I \cap I^{\perp}$ is an ideal in $\mathfrak{g}$ it also follows that

$$\kappa_{I \cap I^{\perp}}(x, y) = \kappa(x, y) = 0 \quad \text{for all } x, y \in I \cap I^{\perp}.$$

By Cartan's criterion $I \cap I^{\perp}$ is a solvable ideal in $\mathfrak{g}$, from which it follows from the previous implication ($4 \Rightarrow 3$) that $I \cap I^{\perp} = 0$. Therefore $\mathfrak{g} = I \oplus I^{\perp}$, where I and $I^{\perp}$ are proper ideals in $\mathfrak{g}$. By Lemma 2.2.14 the Killing form κ is given by the sum of the Killing forms κ_{I_1} and κ_{I_2}. AS κ is non-degenerate it follows that the same goes for κ_{I_1} and κ_{I_2}. Hence by induction hypothesis both I_1 and I_2 are the sum of simple ideals $I_1 = J_1 \oplus \cdots \oplus J_r$ and $I_2 = K_1 \oplus \cdots \oplus K_s$. It follows that

$$\mathfrak{g} = I_1 \oplus I_2 = J_1 \oplus \cdots J_r \oplus K_1 \oplus \cdots \oplus K_s$$

is a decomposition into simple ideals.

($2 \Rightarrow 1$) Follows from $\mathfrak{g} = I_1 \oplus \cdots \oplus I_s \cong I_1 \times \cdots \times I_s$.

($1 \Rightarrow 4$) For each $i = 1, \dots, r$ let $\pi_i \colon \mathfrak{g} \to \mathfrak{g}_i$ be the canonical projection and let $I \trianglelefteq \mathfrak{g}$ be a solvable ideal. Then for any $i = 1, \dots, r$ the image $\pi_i(I) \subseteq \mathfrak{g}_i$ is a solvable ideal. Because $\mathfrak{g}_i$ is simple it follows that $\pi_i(I) = 0$ for every $i = 1, \dots, n$. Hence $I = 0$.

($4 \Rightarrow 5$) This directly follows from the fact that every abelian ideal is solvable.

($5 \Rightarrow 4$) Suppose that there exists a nonzero solvable ideal $I \trianglelefteq \mathfrak{g}$. Then let $i \geq 0$ such that $I^{(i+1)} = 0$ but $I^{(i)} \neq 0$. Then $I^{(i)}$ is a nonzero abelian ideal in $\mathfrak{g}$. □

Corollary 3.1.7. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $\mathfrak{g} = I_1 \oplus \cdots \oplus I_n$ *the decomposition into simple ideals. If* $I \subseteq \mathfrak{g}$ *is any ideal, then* $I = I_{i_1} \oplus \cdots \oplus I_{i_m}$ *for some indices* $1 \leq i_1 < \cdots < i_m \leq n$*.*

Proof. The statement holds if $I = \mathfrak{g}$ or $I = 0$, hence it can be assumed that I is a non-trivial ideal of $\mathfrak{g}$. From the proof of the implication (3 ⇒ 2) it follows that there exists a decomposition $\mathfrak{g} = J_1 \oplus \cdots \oplus J_m$ into simple ideals such that $I = J_1 \oplus \cdots \oplus J_\ell$ for some $1 \leq \ell \leq m$. As such a decomposition is unique up to reordering of the summands by Corollary 3.1.4 the statement follows. □

Corollary 3.1.8. *If $\mathfrak{g}$ is a semisimple Lie algebra and $I \trianglelefteq \mathfrak{g}$ any ideal then both I and $\mathfrak{g}/I$ are also semisimple.*

Corollary 3.1.9. *Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra. Then the map*

$$\mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \kappa(x, \cdot)$$

is an isomorphism of representations of $\mathfrak{g}$.

Proof. The statement follows from Corollary 2.2.4 because κ is non-degenerate. □

Corollary 3.1.10. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\beta\colon \mathfrak{g}\times\mathfrak{g} \to k$ an associative bilinear form. Then β is a scalar multiple of the killing form $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to k$.*

Proof. Because β is associative the map

$$\varphi\colon \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \beta(x, \cdot)$$

is a homomorphism of representations of $\mathfrak{g}$ by Lemma 2.2.3. Because $\mathfrak{g}$ is simple, and therefore also semisimple, the Killing form κ is non-degenerate. Therefore the map

$$\psi\colon \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \kappa(x, \cdot)$$

is an isomorphism of representations of $\mathfrak{g}$ by Corollary 3.1.9. It follows that

$$\alpha := \varphi \circ \psi^{-1}\colon \mathfrak{g} \to \mathfrak{g},$$

is a homomorphism of representations of $\mathfrak{g}$. Because $\mathfrak{g}$ is simple the adjoint representation of $\mathfrak{g}$ is irreducible. By Schur's Lemma α is given by multiplication with a scalar $\lambda \in k$. It follows that $\varphi = \lambda\psi$. □

3.2. Weyl's Theorem on complete reducibility

Lemma 3.2.1. *Let $\mathfrak{g}$ be an n-dimensional semisimple Lie algebra, V a finite dimensional vector space and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$.*

1. *The map $\beta\colon \mathfrak{g}\times\mathfrak{g} \to k$ with $\beta(x, y) := \operatorname{tr}(\rho(x)\rho(y))$ for all $x, y \in \mathfrak{g}$ is an associative and symmetric bilinear form.*

2. *If ρ is a faithful representation then β is non-degenerate.*

3. *Suppose ρ is faithful. Let $x_1, \dots, x_n$ ba a basis of $\mathfrak{g}$ and $x^1, \dots, x^n$ the dual basis with respect to β. Then the map*

$$C_\beta^V : V \to V, \quad v \mapsto C_\beta \cdot v = \sum_{i=1}^n x_i.x^i.v = \sum_{i=1}^n \rho(x_i)\rho(x^i)(v)$$

is a homomorphism of representations with $\operatorname{tr}(C_\beta^V) = \dim_k \mathfrak{g}$.

Proof. 1. This was already remarked in Remark 2.2.11.

2. Because $\operatorname{rad}\beta \trianglelefteq \mathfrak{g}$ is and ρ is an isomorphism onto its image it follows that $\rho(\operatorname{rad}\beta) \trianglelefteq \rho(\mathfrak{g})$ is an ideal with $\operatorname{tr}(xy) = 0$ for all $x, y \in \rho(\mathfrak{g})$. By Cartan's criterion for linear Lie algebras (see Lemma 2.2.25) the ideal $\rho(\operatorname{rad}\beta)$ is solvable. Because $\rho(\mathfrak{g})$ is semisimple it follows that $\rho(\operatorname{rad}\beta) = 0$ and hence $\operatorname{rad}\beta = 0$ because ρ is faithful.

3. That C_β^V is a homomorphism of representations was follows from Corollary 2.3.51 and notice that

$$\operatorname{tr}(C_\beta^V) = \operatorname{tr}\left(\sum_{i=1}^n \rho(x_i)\rho(x^i)\right) = \sum_{i=1}^n \operatorname{tr}(\rho(x_i)\rho(x^i)) = \sum_{i=1}^n \beta(x_i, x^i) = n = \dim \mathfrak{g}.$$

□

Definition 3.2.2. Let V be a faithful representation of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ and $\beta\colon \mathfrak{g} \times \mathfrak{g} \to k$ the associative, symmetric and non-degenerate bilinear form defined by $\beta(x, y) = \operatorname{tr}(\rho(x)\rho(y))$ for all $x, y \in \mathfrak{g}$. Then define

$$C_\mathfrak{g}^V : V \to V, \quad v \mapsto C_\beta \cdot v.$$

Remark 3.2.3. Notice for the situation of Definition 3.2.2 that if $x_1, \dots, x_n$ is a basis of $\mathfrak{g}$ and $x^1, \dots, x^n$ is the dual basis with respect to β then $C_\mathfrak{g}^V$ can also be expressed as

$$C_\mathfrak{g}^V = \sum_{i=1}^n \rho(x_i)\rho(x^i).$$

Remark 3.2.4. Definition 3.2.2 was not given in the lecture, but helps to bake the following easier to read.

Lemma 3.2.5. *Let g be a finite dimensional semisimple k-Lie algebra and let V be a finite dimensional representation of $\mathfrak{g}$. Then $V = \mathfrak{g}V \oplus V^\mathfrak{g}$ for the subrepresentations*

$$\mathfrak{g}V \coloneqq \{x.v \mid x \in \mathfrak{g}, v \in V\} \quad \textit{and} \quad V^\mathfrak{g} \coloneqq \{v \in V \mid x.v = 0 \textit{ for every } x \in \mathfrak{g}\}$$

Proof. If $V = U_1 \oplus \dots \oplus U_n$ is a decomposition into subrepresentations of $\mathfrak{g}$ then $\mathfrak{g}V = \mathfrak{g}U_1 \oplus \dots \oplus \mathfrak{g}U_n$ and $V^\mathfrak{g} = U_1^\mathfrak{g} \oplus \dots \oplus U_n^\mathfrak{g}$. Hence it sufficies to show the statement for indecomposable representations. Hence suppose that V is indecomposable. It can also be assumed that $V \neq 0$.

By replacing $\mathfrak{g}$ with $\rho(\mathfrak{g})$ it can be assumed w.l.o.g. that V is a faithful representation of $\mathfrak{g}$. Notice that $\rho(\mathfrak{g})$ is also semisimple by Corollary 3.1.8.

Now let $C \coloneqq C_{\mathfrak{g}}^V \colon V \to V$ be the endomorphism of V defined as in Lemma 3.2.1. Because C is a homomorphism of representations of $\mathfrak{g}$ and V is finite dimensional it follows that V decomposes into the generalized eigenspaces of C. Because V is indecomposable by assumption it follows that V is the generalized eigenspace of C with respect to some $\lambda \in k$ (here it is used that $V \neq 0$ by assumption). Notice that by Lemma 3.2.1

$$\dim_k \mathfrak{g} = \operatorname{tr}(C) = (\dim_k V) \cdot \lambda.$$

If $\lambda = 0$ then it follows that $\mathfrak{g} = 0$ (this can also be seen directly because $\mathfrak{g}$ acts faithful by assumption). Then $V = V^{\mathfrak{g}}$ and the statement holds.

Suppose otherwise that $\lambda \neq 0$. Then $\det C = \lambda^{\dim_k V} \neq 0$, hence C is actually an automorphism of V, hence $C(V) = V$. As C is given by multiplication with a Casimir element $C_\beta \in \mathcal{U}(\mathfrak{g})$ it follows that

$$\mathfrak{g}V = \{x.v \mid x \in \mathfrak{g}, v \in V\} = \{x \cdot v \mid x \in \mathcal{U}(\mathfrak{g}) \mid V\} \supseteq \{C_\beta \cdot v \mid v \in V\} = C(V) = V,$$

hence $\mathfrak{g}V = V$. □

Theorem 3.2.6 (Weyl)**.** *Let V be finite dimensional representation of a finite dimensional semisimple Lie algebra $\mathfrak{g}$. Then V is completely reducible.*

Proof. It sufficies to show that any subrepresentation $U \subseteq V$ has a direct complement which is again a subrepresentation, i.e. that there exists a subrepresentation $W \subseteq V$ with $V = U \oplus W$.

By Lemma 3.2.5 there exists a short exact sequence

$$0 \to \mathfrak{g}\operatorname{Hom}_k(V, U) \to \operatorname{Hom}_k(V, U) \xrightarrow{\pi_V} \operatorname{Hom}_k(V, U)^{\mathfrak{g}} \to 0$$

as well as a short exact sequence

$$0 \to \mathfrak{g}\operatorname{Hom}_k(U, U) \to \operatorname{Hom}_k(U, U) \xrightarrow{\pi_U} \operatorname{Hom}_k(U, U)^{\mathfrak{g}} \to 0.$$

The restriction map $\rho \colon \operatorname{Hom}_k(V, U) \to \operatorname{Hom}_k(U, U), f \mapsto f|_U = f \circ \iota$ is surjective, where $\iota \colon U \hookrightarrow V$ denotes the canonical inclusion (notice that it a map between the k-linear maps). It is a homomorphism of representations of $\mathfrak{g}$, hence induces a map $\bar{\rho} \colon \operatorname{Hom}_k(V, U)^{\mathfrak{g}} \to \operatorname{Hom}_k(U, U)^{\mathfrak{g}}$ by restriction. The maps so far fit in the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_k(V, U) & \xrightarrow{\ \rho\ } & \operatorname{Hom}_k(U, U) \\ \Big\downarrow \pi_V & & \Big\downarrow \pi_U \\ \operatorname{Hom}_k(V, U)^{\mathfrak{g}} & \xrightarrow[\ \bar{\rho}\ ]{} & \operatorname{Hom}_k(U, U)^{\mathfrak{g}} \end{array}$$

Because π_U and ρ are surjective it follows that $\bar{\rho}$ is also surjective. Thus there exists some $\pi \in \mathrm{Hom}_k(V,U)^{\mathfrak{g}}$ with $\mathrm{id}_U = \bar{\rho}(\pi) = \pi \circ \iota$.

Notice that $\mathrm{Hom}_k(V,U)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V,U)$ and $\mathrm{Hom}_k(U,U)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(U,U)$ as already seen in Remark 1.2.13. Hence $\mathrm{id}_U = \pi \circ \iota$ is a statement about homomorphism of representations of $\mathfrak{g}$, i.e. homomorphisms of $\mathcal{U}(\mathfrak{g})$-modules. It follows that π is a retraction in the category of $\mathcal{U}(\mathfrak{g})$-modules, which is why $V = U \oplus \ker \pi$ with $\ker \pi \subseteq V$ being a subrepresentation of $\mathfrak{g}$. □

Remark 3.2.7. As a representation V of a Lie algebra $\mathfrak{g}$ is completely reducible if and only if it is semisimple as an $\mathcal{U}(\mathfrak{g})$-module one has the usual equivalent definitions of complete reducibility, i.e. the following are equivalent:

1. V is a completely reducible, i.e. V is the direct sum of irreducible subrepresentations.

2. V is the direct sum of irreducible subrepresentations.

3. Every submodule $U \subseteq V$ has a direct summand which is also a subreprentation.

Remark 3.2.8. Weyl's theorem does not necessarily hold for infine dimensional representations. To see this take $k[X]$ as a representation of $\mathfrak{sl}_2(k)$ via

$$\mathfrak{sl}_2(k) \to \mathfrak{gl}(k[X]), \quad e \mapsto \frac{\mathrm{d}}{\mathrm{d}X}, \quad h \mapsto -2X\frac{\mathrm{d}}{\mathrm{d}X}, \quad f \mapsto -x^2\frac{\mathrm{d}}{\mathrm{d}X}$$

which was already shown in Examples 1.2.5 to define a representation of $\mathfrak{sl}_2(k)$. Notice that $k \subseteq k[X]$ is a subrepresentation. If $U \subseteq k[X]$ is a subrepresentation and $P \in U$ a polynomial of degree n then by applying e it turns out that U contains of polynomial of degree d for every $d \leq n$, which is why U contains $1, X, \dots, X^n$ (here it is used that $\mathrm{char}\, k = 0$). Moreover, if $X \in U$ then by applying f it turns out that $X^d \in U$ for every $d \geq 1$. Hence if U contains a non constant polynomial then already $X^d \in U$ for every $d \in \mathbb{N}$ and therefore $U = k[X]$.

So 0, k and $k[X]$ are the only subrepresentations of $k[X]$. in particular $k[X]$ is not completely reducible.

3.3. Finite dimensional representations of $\mathfrak{sl}_2(k)$

Recall that the standard basis (e,h,f) of $\mathfrak{sl}_2(k)$ is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and that this basis satisfies the relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Definition 3.3.1. Let V be a representation of $\mathfrak{sl}_2(k)$. For any $\lambda \in k$ let

$$V_\lambda := \{v \in V \mid h.v = \lambda v\}$$

be the *weight space* of V with respect to λ. An element $\lambda \in k$ is called a *weight* of V if $V_\lambda \neq 0$.

Lemma 3.3.2. *If V is a representation of $\mathfrak{g}$ then*

$$e.V_\lambda \subseteq V_{\lambda+2} \quad \text{and} \quad f.V_\lambda \subseteq V_{\lambda-2} \quad \text{for every } \lambda \in k.$$

Proof. Let $v \in V_\lambda$. Then

$$h.(e.v) = e.h.v + [h,e].v = \lambda e.v + 2e.v = (\lambda+2)e.v$$

and

$$h.(f.v) = h.f.v + [h,f].v = \lambda h.v - 2f.v = (\lambda-2)f.v.$$ □

Example 3.3.3. Let e_1, e_2 be the standard basis of $V := k^2$, which is the natural representation of $\mathfrak{sl}_2(k)$. Then $h.e_1 = e_1$ and $h.e_2 = -e_2$, so $V_1 = ke_1$ and $V_{-1} = ke_2$ with $V = V_{-1} \oplus V_1$. That $e.V_{-1} = V_1$ and $f.V_1 = V_{-1}$ can then by seen by a glance at the following:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

3.3.1. Finite dimensional irreducible representations

Theorem 3.3.4 (Classification of finite dimensional irreducible representations of $\mathfrak{sl}_2(k)$)**.** *For every $n \in \mathbb{N}$ there exists up to isomorphism a unique $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(k)$. More explicitely:*

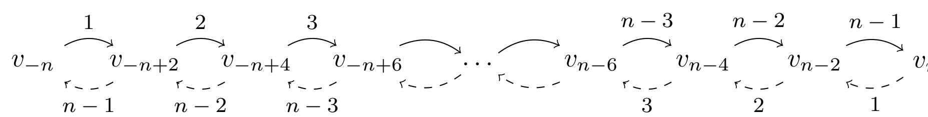
1. *For every $n \in \mathbb{N}$ there exists an $(n+1)$-dimensional irreducible representation $V^{(n)}$ of $\mathfrak{sl}_2(k)$.*

2. *Let V be an $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(k)$ for some $n \in \mathbb{N}$. Then*

   $$V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$$

   and V_i is one-dimensional for every $i = -n, -n+2, \ldots, n-2, n$. In particular all occuring weights are integral and the heighest weight is n.

   More explicitely there exists a basis $v_{-n}, v_{-n+2}, \ldots, v_{n-2}, v_n$ of V with $v_i \in V_i$ for every $i = -n, -n+2, \ldots, n-2, n$, with respect to which the actions of e and f are given as in the following diagram, where the dashed arrows represent the action of f.



In particular

$$e.V_i = \begin{cases} V_{i+2} & \text{for } i = -n, -n+2, \dots, n-2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f.V_i = \begin{cases} V_{i-2} & \text{if } i = -n, \dots, n-2, n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. 1. Let $\mathfrak{sl}_2(k)$ act on $V := k[x,y]$ via

$$\mathfrak{sl}_2(k) \to \mathfrak{gl}(V), \quad e \mapsto y\frac{\mathrm{d}}{\mathrm{d}x}, \quad h \mapsto y\frac{\mathrm{d}}{\mathrm{d}y} - x\frac{\mathrm{d}}{\mathrm{d}x}, \quad f \mapsto x\frac{\mathrm{d}}{\mathrm{d}y}.$$

It was already shown in Examples 1.2.5 that this defines a representation of $\mathfrak{sl}_2(k)$. Let $V^{(n)} \subseteq V$ be the linear subspace consisting of the homogeneous polynomials of degree n, i.e.

$$V^{(n)} = \operatorname{span}_k \left(x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\right).$$

Then $V^{(n)}$ is an $(n+1)$-dimensional subrepresentation of V.

Let $U \subseteq V^{(n)}$ be a nonzero subrepresentation. If $p \in U$ is a nonzero polynomial then by applying f often enough it follows that $x^n \in U$, from which it follows from applying e often enough that $x^{n-i}y^i \in U$ for every $i = 0, \dots, n$. Hence $U = V$. So $V^{(n)}$ is an irreducible representation of $\mathfrak{sl}_2(k)$.

2. Because $V \neq 0$ is finite dimensional and k is algebraically closed there exists some $\lambda \in k$ with $V_\lambda \neq 0$. Because v is finite dimensional λ can be choosen such that $V_{\lambda-2} = 0$. Let $w \in V_\lambda$ with $w \neq 0$. Set

$$w_i := e^i.w \quad \text{for every } i \in \mathbb{N}.$$

Claim. *a) $h.w_i = (\lambda + 2i)w_i$ for every $i \in \mathbb{N}$.*

b) $f.w_0 = 0$ and $f.w_{i+1} = -(i+1)(\lambda+i)w_i$ for every $n \in \mathbb{N}$.

Proof. a) This follows from $w \in V_\lambda$ and Lemma 3.3.2.

b) From $w_0 = w \in V_\lambda$ and Lemma 3.3.2 it follows that $f.w_0 \in V_{\lambda-2}$. Because $V_{\lambda-2} \neq 0$ it follows that $f.w_0 = 0$.

The second formula will be shown by induction over $i \in \mathbb{N}$. It holds for $i = 0$ because

$$f.w_1 = f.e.w_0 = [f,e].w_0 - e.f.w_0 = -h.w_0 = -\lambda w_0.$$

Now let $i > 0$ and suppose the formula holds for $i-1$. Then

$$\begin{aligned} f.w_{i+1} = f.e.w_i &= [f,e].w_i + e.f.w_i = -h.w_i + e.f.w_i \\ &= -(\lambda+2i)w_i - i(\lambda+i-1)e.w_{i-1} \\ &= (-\lambda - 2i - i\lambda - i^2 + i)w_i = -(i+1)(\lambda+i)w_i. \end{aligned}$$ □

Because $w_i \in V_{\lambda+2i}$ for every $i \in \mathbb{N}$ with $w_0 = v \neq 0$ and V is finite dimensional it follows that there exists a maximal $m \in \mathbb{N}$ such that $w_0, \dots, w_m$ are nonzero but $w^{m+1} = 0$. By the previous claim $\operatorname{span}_k(w_0, \dots, w_m)$ is a subrepresentation of V. Because V is irreducible it follows that $V = \operatorname{span}_k(w_0, \dots, w_m)$. Because $w_0, \dots, w_m$ are linearly independent it follows that $w_0, \dots, w_m$ is a basis of V.

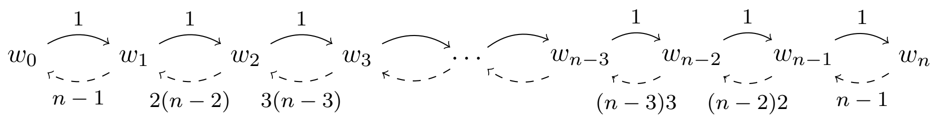
As V is $(n+1)$-dimensional it follows that $m = n$. By the claim

$$0 = f.w_{n+1} = -n(\lambda + n).w_n$$

and therefore $\lambda = -n$. Because $w_i \in V_{\lambda+2i} = V_{-n+2i}$ for every $i \in \mathbb{N}$ it follows that

$$V = kw_0 \oplus \cdots \oplus kw_n = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$$

with V_i being one-dimensional for every $i = -n, -n+2, \dots, n-2, n$. From the definition of $w_0, \dots, w_n$ and the claim it follows that the actions of e and f are given as in the following diagram, where the dashed arrows represent the action of f.



The desired basis $v_1, \dots, v_n$ can now be defined as

$$v_{-n+2i} := \frac{w_i}{i!} \quad \text{for every } i = 0, 1, \dots, n+1. \qquad \square$$

Examples 3.3.5. 1. The one-dimensional irreducible representation of $\mathfrak{sl}_2(k)$ can be realized by $\mathfrak{sl}_2(k)$ acting trivially on k (or any one-dimensional vector space for that matter.)

2. The two-dimensional irreducible representation of $\mathfrak{sl}_2(k)$ can be realized as the natural representation of $\mathfrak{sl}_2(k)$. The detailed calculations were already shown in Example 3.3.3.

3. Let $V = \mathfrak{sl}_2(k)$ be the adjoint representation of $\mathfrak{sl}_2(k)$. Because $\mathfrak{sl}_2(k)$ is simple it follows that V is irreducible and $V = V_{-2} \oplus V_0 \oplus V_2$ with $V_{-2} = kf$, $V_0 = kh$ and $V_2 = ke$.

3.3.2. Arbitrary finite dimensional representations

Example 3.3.6. The map

$$\phi \colon \mathfrak{sl}_2(k) \to \mathfrak{sl}_3(k), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

is a homomorphism of Lie algebras. Therefore $V := \mathfrak{sl}_3(k)$ can be made into a representation of $\mathfrak{sl}_2(k)$ via

$$\rho \colon \mathfrak{sl}_2(k) \xrightarrow{\phi} \mathfrak{sl}_3(k) \xrightarrow{\text{ad}} \mathfrak{gl}(\mathfrak{sl}_3(k)) = \mathfrak{gl}(V).$$

Then h acts on V by $\operatorname{ad}(H)$ for

$$H := \phi(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Notice that this representation is not irreducible because $\phi(\mathfrak{sl}_2(k))$ is a non-trivial subrepresentation. However, by Weyl's theorem V decomposes into the direct sum of irreducible subrepresentations and it will be shown how to do so:

Let $e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}, h_1, h_2$ be the basis of V with

$$h_1 := \operatorname{diag}(1, -1, 0) \quad \text{and} \quad h_2 := \operatorname{diag}(0, 1, -1).$$

Notice that

$$[\operatorname{diag}(a_1, a_2, a_3), e_{ij}] = (a_i - a_j)e_{ij} \quad \text{for all } a_1, a_2, a_3 \in k \text{ and } i, j = 1, \dots, 3.$$

It follows that

$$e_{21} \in V_{-2}, \quad e_{23}, e_{31} \in V_{-1}, \quad h_1, h_2 \in V_0, \quad e_{13}, e_{32} \in V_1, \quad e_{12} \in V_2.$$

So $V = \bigoplus_{i\in\mathbb{Z}} V_i$ with dimensions

$$\dim V_i = \begin{cases} 1 & \text{if } i = -2, 2, \\ 2 & \text{if } i = -1, 0, 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Suppose that $V = W^1 \oplus W^2$ for two four-dimensional irreducible subrepresentations W^1 and W^2. Then

$$W^1 = W^1_{-3} \oplus W^1_{-1} \oplus W^1_1 \oplus W^1_3 \quad \text{and} \quad W^2 = W^2_{-3} \oplus W^2_{-1} \oplus W^2_1 \oplus W^2_3.$$

and it follows that

$$V = \underbrace{W^1_{-3} \oplus W^2_{-3}}_{=V_{-3}} \oplus \underbrace{W^1_{-1} \oplus W^2_{-1}}_{=V_{-1}} \oplus \underbrace{W^1_1 \oplus W^2_1}_{=V_1} \oplus \underbrace{W^1_3 \oplus W^2_3}_{=V_3},$$

contradicting (1).

To emphasize the chosen approach suppose that there exists a decomposition $V = W^{0,1} \oplus W^{0,2} \oplus W^{0,3} \oplus W^{1,1} \oplus W^{2,1}$ into irreducible subrepresentations with $\dim W^{0,1} = \dim W^{0,2} = \dim W^{0,3} = 1$, $\dim W^{1,1} = 2$ and $\dim W^{1,1} = 3$. Then

$$\begin{aligned} V &= W^{0,1} \oplus W^{0,2} \oplus W^{0,3} \oplus W^{1,1} \oplus W^{2,1} \\ &= \underbrace{W^{2,1}_{-2}}_{=V_{-2}} \oplus \underbrace{W^{1,1}_{-1}}_{=V_{-1}} \oplus \underbrace{W^{0,1}_0 \oplus W^{0,2}_0 \oplus W^{0,3}_0 \oplus W^{2,1}_0}_{=V_0} \oplus \underbrace{W^{1,1}_1}_{=V_1} \oplus \underbrace{W^{2,1}_1}_{=V_2}, \end{aligned}$$

also contradicting (1).

The above observations can easily be generalized: Let $V = \bigoplus_{d\in\mathbb{N}} \bigoplus_{j=1}^{\nu_d} W^{d,j}$ be a decomposition into irreducible subrepresentations with $W^{d,j}$ being $(d+1)$-dimensional, i.e. having heighest weight d. As in the previous examples it follows that $V_i = \bigoplus_{p\in\mathbb{N},d=|i|+2p} \bigoplus_{j=1}^{\nu_d} W_i^{d,j}$ for every $i \in \mathbb{Z}$ and therefore $\dim V_i = \sum_{p\in\mathbb{N},d=|i|+2p} \nu_d$. Hence

$$\nu_d = \dim V_d - \dim V_{d+2} \quad \text{for every } d \in \mathbb{N}.$$

With this it follows from (1) that $\nu_0 = 1$, $\nu_1 = 2$ and $\nu_2 = 1$.

The three-dimensional irreducible subrepresentation is given by

$$W^{2,1} := \phi(\mathfrak{sl}_2(k)) = \operatorname{span}_k(e_{12}, h_1, e_{21})$$

The two two-dimensional irreducible subrepresentations $W^{1,1}$ and $W^{1,2}$ must satisfy

$$W^{1,1} \oplus W^{1,2} = V_{-1} \oplus V_1 = \operatorname{span}_k(e_{23}, e_{31}, e_{13}, e_{32}).$$

They then can be choosen as $W^{1,1} := \operatorname{span}_k(e_{23}, e_{13})$ and $W^{1,2} := \operatorname{span}_k(e_{31}, e_{32})$. That these are indeed subrepresentations follows from direct calculation. To find the remaining one-dimensional irreducible subrepresentation notice that

$$e.h_1 = -2e_{12}, \quad h.h_1 = 0, \quad f.h_1 = 2e_{21},$$

and

$$e.h_2 = e_{12}, \quad h.h_2 = 0, \quad f.h_2 = -e_{21}.$$

Hence $\mathfrak{sl}_2(k)$ acts trivially on the one-dimensional linear subspace $W^{0,1} := k(h_1+2h_2)$, which is why it is a subrepresentation.

Theorem 3.3.7. *Let V be a finite dimensional representation of $\mathfrak{sl}_2(k)$.*

1. $V = \bigoplus_{d\in\mathbb{N}} \bigoplus_{j=1}^{\nu_d} W^{d,j}$ *where* $W^{d,j} \subseteq V$ *is an irreducible* $(d+1)$*-dimensional subrepresentation for all* $d \in \mathbb{N}$ *and* $j = 1, \dots, \nu_d$.

2. $V = \bigoplus_{i\in\mathbb{Z}} V_i$ *with*

$$V_i = \bigoplus_{\substack{p\in\mathbb{N}\\ d=|i|+2p}} \bigoplus_{j=1}^{\nu_d} W_i^{d,j} \quad \text{for every } i \in \mathbb{Z} \tag{2}$$

and $\dim V_i = \sum_{p\in\mathbb{N},d=|i|+2p} \nu_d$ *for every* $i \in \mathbb{Z}$. *In particular all occuring weights are integral.*

3. *The numbers* ν_d *for* $d \in \mathbb{N}$ *are unique with* $\nu_d = \dim V_d - \dim V_{d+2}$ *for every* $d \in \mathbb{N}$.

4. $\dim V_i = \dim V_{-i}$ *for every* $i \in \mathbb{Z}$.

Proof. That V decomposes into the direct sum of irreducible subrepresentations follows directly from Weyl's theorem. From the classification of finite dimensional irreducible representations of $\mathfrak{sl}_2(k)$ it follows that

$$W^{d,j} = W^{d,j}_{-d} \oplus W^{d,j}_{-d+2} \oplus \cdots \oplus W^{d,j}_{d-2} \oplus W^{d,j}_{d} = \bigoplus_{p=0}^{d} W^{d,j}_{-d+2p}$$

for every $d \in \mathbb{N}$ and $j = 1, \ldots, \nu_d$. It follows that

$$V = \bigoplus_{d\in\mathbb{N}} \bigoplus_{j=1}^{\nu_d} W^{d,j} = \bigoplus_{d\in\mathbb{N}} \bigoplus_{j=1}^{\nu_d} \bigoplus_{p=0}^{d} W^{d,j}_{-d+2p}.$$

This shows that $V_i = \bigoplus_{i\in\mathbb{Z}} V_i$. Formula (2) follows from reordering the summands. It follows that

$$\dim V_i = \dim \bigoplus_{\substack{p\in\mathbb{N} \\ d=|i|+2p}} \bigoplus_{j=1}^{\nu_d} W^{d,j}_i = \sum_{\substack{p\in\mathbb{N} \\ d=|i|+2p}} \sum_{j=1}^{\nu_d} \underbrace{\dim W^{d,j}_i}_{=1} = \sum_{\substack{p\in\mathbb{N} \\ d=|i|+2p}} \nu_d.$$

As a direct consequence it follows that $\dim V_i = \dim V_{-i}$ for every $i \in \mathbb{Z}$ and

$$\dim V_d - \dim V_{d+2} = \sum_{\substack{p\in\mathbb{N} \\ d'=i+2p}} \nu_{d'} - \sum_{\substack{p\in\mathbb{N} \\ d'=i+2+2p}} \nu_{d'} = \nu_d \quad \text{for every } d \in \mathbb{N}.$$ □

3.3.3. The Clebsch–Gordan decomposition

Example 3.3.8. For every $n \in \mathbb{N}$ let $V^{(n)}$ be the $(n+1)$-dimensional irreducible representation of V and abbreviate $V := V^{(3)}$ and $W := V^{(4)}$. Then

$$V \otimes W \cong V^{(7)} \otimes V^{(5)} \otimes V^{(3)} \otimes V^{(1)}. \tag{3}$$

To see this notice that for $i, j \in \mathbb{Z}$ with $v \in V_\lambda$ and $w \in W_\lambda$ it follows that

$$h.(v \otimes w) = (h.v) \otimes w + v \otimes (h.w) = iv \otimes w + jv \otimes w = (i+j)v \otimes w$$

and therefore $V_i \otimes W_j \subseteq (V \otimes W)_{i+j}$. It follows that

$$\begin{aligned}
(V \otimes W)_{-7} &= V_{-3} \otimes W_{-4}, \\
(V \otimes W)_{-5} &= (V_{-3} \otimes W_{-2}) \oplus (V_{-1} \oplus W_{-4}), \\
(V \otimes W)_{-3} &= (V_{-3} \otimes W_{0}) \oplus (V_{-1} \otimes W_{-2}) \oplus (V_{1} \otimes W_{-4}), \\
(V \otimes W)_{-1} &= (V_{-3} \otimes W_{2}) \oplus (V_{-1} \otimes W_{0}) \oplus (V_{1} \otimes W_{-2}) \oplus (V_{3} \otimes W_{-4}), \\
(V \otimes W)_{1} &= (V_{-3} \otimes W_{4}) \oplus (V_{-1} \otimes W_{2}) \oplus (V_{1} \otimes W_{0}) \oplus (V_{3} \otimes W_{-2}), \\
(V \otimes W)_{3} &= (V_{-1} \otimes W_{4}) \oplus (V_{1} \otimes W_{2}) \oplus (V_{3} \otimes W_{0}), \\
(V \otimes W)_{5} &= (V_{1} \otimes W_{4}) \oplus (V_{3} \otimes W_{2}), \\
(V \otimes W)_{7} &= V_{3} \otimes W_{4}.
\end{aligned}$$

with each of the summands being one-dimensional. Therefore the weights spaces of $V \otimes W$ have the following dimensions:

$i \in \mathbb{Z}$	-7	-5	-3	-1	1	3	5	7	otherwise
$\dim(V \otimes W)_i$	1	2	3	4	4	3	2	1	0

Formula (3) follows from this by Theorem 3.3.7.

Proposition 3.3.9. *For all $n \in \mathbb{N}$ let $V^{(n)}$ denote the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(k)$. Then*

$$V^{(n)} \otimes V^{(m)} \cong V^{(n+m)} \oplus V^{(n+m-2)} \oplus \cdots \oplus V^{(|n-m|)} \quad \text{for all } n, m \in \mathbb{N}.$$

Proof. It can be assummed w.l.o.g. that $n \geq m$. Abbreviate $V := V^{(n)}$ and $W := W^{(m)}$. Then $V = \bigoplus_{p=0}^{n} V_{-n+2p}$ and $W = \bigoplus_{q=0}^{m} W_{-m+2q}$ and therefore

$$V \otimes W = \bigoplus_{p=0}^{n} \bigoplus_{q=0}^{m} \underbrace{V_{-n+2p} \otimes W_{-m+2q}}_{\subseteq (V \otimes W)_{-(n+m)+2(p+q)}} .$$

Because $V \otimes W$ decomposes into weight spaces it follows that

$$(V \otimes W)_i = \bigoplus_{\substack{j_1=-n,-n+2,\dots,n-2,n \\ j_2=-m,-m+2\dots,m-2,m \\ p+q=i}} V_p \otimes W_q \quad \text{for every } i \in \mathbb{Z}.$$

Therefore $\dim(V \otimes W)_i$ is the number of solutions of the equation

$$p+q=i \quad \text{with} \quad p \in \{-n,-n+2,\dots,n-2,n\}, q \in \{-m,-m+2,\dots,m-2,m\}. \quad (4)$$

To count these solutions notice that (4) has no solution for $i < -n-m$ or $i > n+m$. Also notice that $p+q \equiv n+m \pmod{()2}$ in (4). So it is enough to count the solutions of (4) for $i = -(n+m), -(n+m)+2, \dots, n+m-2, n+m$. The solutions are given in the table in Figure 3.1 (page 67). This results in the following dimensions:

$i \in \mathbb{Z}$	$-n-m$	$-n-m+2$	$\cdots$	$-n+m$	$\cdots$	$n-m$	$\cdots$	$n+m-2$	$n+m$
$\dim(V \otimes W)_i$	1	2	$\cdots$	$m+1$	$\cdots$	$m+1$	$\cdots$	2	1

From this it follows from Theorem 3.3.7 that

$$V \otimes W \cong V^{(n+m)} \oplus V^{(n+m-2)} \oplus \cdots \oplus V^{(n-m)}.$$ □

We also give another proof, taken from [**Lectures_on_sl2_modules**].

Proof. It can be assummed w.l.o.g. that $n \geq m$. The formula can then be shown by induction on m. As $\mathfrak{sl}_2(k)$ acts trivially on the one-dimensional representation $V^{(0)} \cong \mathbb{C}$ it follows that the map $V^{(n)} \to V^{(n)} \otimes V^{(0)}, v \mapsto v \otimes 1$ is an isomorphism representations. This shows the formula for $m = 0$.

To show the formula for $m = 1$ first notice the following:

i	solutions (p,q) for $p+q=i$	number
$n+m$	(n,m)	1
$n+m-2$	$(n,m-2),(n-2,m),$	2
$n+m-4$	$(n,m-4),(n-2,m-2),(n-4,m),$	3
$\vdots$	$\vdots$	$\vdots$
$n-m$	$(n,-m),(n-2,-m+2),\dots,(n-2m,m)$	$m+1$
$n-m-2$	$(n-2,-m),(n-4,-m+2),\dots,(n-2m-2,m)$	$m+1$
$\vdots$	$\vdots$	$\vdots$
$-n+m+2$	$(-n+2m+2,-m),(-n+2m,-m+2),\dots,(-n+2,m)$	$m+1$
$-n+m$	$(-n+2m,-m),(-n+2m-2,-m+2),\dots,(-n,m)$	$m+1$
$\vdots$	$\vdots$	$\vdots$
$-n-m+4$	$(-n+4,-m),(-n+2,-m+2),(-n,-m+4)$	3
$-n-m+2$	$(-n+2,-m),(-n,-m+2)$	2
$-n-m$	$(-n,-m)$	1

Figure 3.1.: Solutions for counting dimensions of weight spaces.

Claim. *Let V be a finite dimenisonal representation of $\mathfrak{sl}_2(k)$. Suppose there exist a nonzero $v\in V$ with $h.v=rv$ and $e.v=0$ for $r\in\mathbb{N}$. Then V contains a subrepresentation which is isomorphic to $V^{(r)}$.*

Proof. Let $V=\bigoplus_{d\in\mathbb{N}}\bigoplus_{j=1}^{\nu_d}V^{d,j}$ be a decomposition into irreducible subrepresentations with $V^{d,j}$ having heighest weight d for every $d\in\mathbb{N}$ and $j=1,\dots,\nu_d$. Then $v\in V_r=\bigoplus_{p\in\mathbb{N},d=r+2p}\bigoplus_{j=0}^{\nu_d}V_r^{d,j}$. Suppose that V contains no submodule which is isomorphic to $V^{(r)}$. Then $\nu_r=0$ and therefore $V_r=\bigoplus_{p\geq 1,d=r+2p}\bigoplus_{j=1}^{\nu_d}V_r^{d,j}$. Because $e.V_r^{d,j}=V_{r+2}^{d,j}$ for every $d>r$ and $j=1,\dots,\nu_d$ it follows that e maps $V_r=\bigoplus_{p\geq 1,d=r+2p}\bigoplus_{j=1}^{\nu_d}V_r^{d,j}$ isomorphically into $\bigoplus_{p\geq 1,d=r+2p}\bigoplus_{j=1}^{\nu_d}V_{r+2}^{d,j}\subseteq V_{r+2}$. Because $e.v=0$ it follows that $e=0$, contradicting the assumption that v is nonzero. ☐

Let $v_{-n},v_{-n+2},\dots,v_{n-2},v_n$ be a basis of $V^{(n)}$ with $v_i\in V_i$ for every i and $e.v_{n-2}=v_n$ (notice that $n\geq m=1$, so v_{n-2} is well-defined). Similarly and w_{-1},w_1 a basis of $V^{(1)}$ with $w_j\in W_j$ for $j=-1,1$. Then $v_n\otimes w_1\in V^{(n)}\otimes V^{(1)}$ with $h.(v_n\otimes w_1)=(n+1)v_n\otimes w_1$ and $e.(v_n\otimes w_1)=0$. By the previous claim $V^{(n)}\otimes V^{(1)}$ contains a subrepresentation $W^1\cong V^{(n+1)}$. Similarly $x\coloneqq v_{n-2}\otimes w_1-v_n\otimes w_2\in V^{(n)}\otimes V^{(1)}$ with $e.x=0$ (here it is used that $e.v_{n-2}=v_n$) and $h.x=(n-1)x$. By the claim V contains another subrepresentation $W^2\cong V^{(n-1)}$. Because W^1 and W^2 are irreducible and not equal it follows that $W^1\cap W^2=0$. Because

$$\dim W^1+\dim W^2=2n+2=\left(\dim V^{(n)}\right)\cdot\left(\dim V^{(1)}\right)=\dim\left(V^{(n)}\otimes V^{(1)}\right)$$

it follows that $V=W^1\oplus W^2\cong V^{(n+1)}\oplus V^{(n-1)}$. This shows the formula for $m=1$.

Suppose that $m \geq 2$ and the statement holds for $0, 1, \ldots, m-1$. Then on the one hand

$$\begin{aligned} V^{(n)} \otimes V^{(m-1)} \otimes V^{(1)} &\cong V^{(n)} \otimes \left(V^{(m)} \oplus V^{(m-2)}\right) \cong V^{(n)} \otimes V^{(m)} \oplus V^{(n)} \otimes V^{(m-2)} \\ &\cong V^{(n)} \otimes V^{(m)} \oplus \left(V^{(n+m-2)} \oplus V^{(n+m-4} \oplus \cdots \otimes V^{(n-m+2)}\right) \end{aligned}$$

while on the other hand

$$\begin{aligned} &V^{(n)} \otimes V^{(m-1)} \otimes V^{(1)} \\ \cong &\left(V^{(n+m-1)} \oplus V^{(n+m-3)} \oplus \cdots \oplus V^{n-m+1}\right) \otimes V^{(1)} \\ \cong &\left(V^{(n+m-1)} \otimes V^{(1)}\right) \oplus \left(V^{(n+m-3)} \otimes V^{(1)}\right) \oplus \cdots \oplus \left(V^{(n-m+1)} \otimes V^{(1)}\right) \\ \cong &\left(V^{(n+m)} \oplus V^{(n+m-2)}\right) \oplus \left(V^{(n+m-2)} \oplus V^{(n+m-4)}\right) \oplus \cdots \oplus \left(V^{(n-m+2)} \oplus V^{(n-m)}\right) \\ \cong &\left(V^{(n+m)} \oplus V^{(n+m-2)} \oplus \cdots \oplus V^{(n-m)}\right) \\ &\oplus \left(V^{(n+m-2)} \oplus V^{(n+m-4)} \oplus \cdots \oplus V^{(n-m+2)}\right). \end{aligned}$$

By the uniqueness of the decomposition of $V^{(n)} \otimes V^{(m-1)} \otimes V^{(1)}$ into irreducible subrepresentations is follows that

$$V^{(n)} \otimes V^{(m)} \cong V^{(n+m)} \oplus V^{(n+m-2)} \oplus \cdots \oplus V^{(n-m)},$$

which finishes the proof. □

Remark 3.3.10. [**Lectures_on_sl2_modules**] has a pretty cool cover and we advise the reader to check it out.

3.4. The abstract Jordan decomposition

In this section the concrete Jordan decomposition (Theorem 2.2.17) is generalized to finite dimensional semisimple Lie algebras. The approach taken is mostly from [**Humphreys**] with some inspiration from the lecture.

Lemma 3.4.1. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a finite dimensional semisimple linear Lie algebra. Then $\mathfrak{g}$ contains the semisimple and nilpotent parts of all its elements.*

Proof. It sufficies to show the statement for $V \neq 0$ because otherwise $\mathfrak{g} = \mathfrak{gl}(V) = 0$.

Let $N \coloneqq N_{\mathfrak{gl}(V)}(\mathfrak{g})$. This is a Lie subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{g}$. Let $x \in \mathfrak{g}$. Because $\operatorname{ad}(x)(\mathfrak{g}) \subseteq \mathfrak{g}$ it follows that $\operatorname{ad}(x_s)(\mathfrak{g}) \subseteq \mathfrak{g}$ and $\operatorname{ad}(x_n)(\mathfrak{g}) \subseteq \mathfrak{g}$ with the abbreviation $\operatorname{ad} = \operatorname{ad}_{\mathfrak{gl}(V)}$. From Lemma 2.2.21 it follows that $\operatorname{ad}(x)_s = \operatorname{ad}(x_s)$ and $\operatorname{ad}(x)_n = \operatorname{ad}(x_n)$ where $x = x_s + x_n$ is the concrete Jordan decomposition of x. Hence $x_s, x_n \in N$.

Notice that $\mathfrak{g} \subsetneq N$ because $\mathrm{id}_V \in N$ but $\mathrm{id}_V \notin \mathfrak{g}$ since $Z(\mathfrak{g}) \neq 0$. The idea is to now shrink N down to $\mathfrak{g}$ by adding additional restraints while maintaining the property that the semisimle and nilpotent parts of all element of $\mathfrak{g}$ are contained.

For any $\mathfrak{g}$-subrepresentation $W \subseteq V$ let

$$\mathfrak{g}_W := \{x \in \mathfrak{gl}(V) \mid x(W) \subseteq W, \operatorname{tr} x|_W = 0\};$$

notice that this is a Lie subalgebra of $\mathfrak{gl}(V)$. Also notice that together with Corollary 3.1.3 it follows that $\mathfrak{g} \subseteq \mathfrak{g}_W$ for every $\mathfrak{g}$-subrepresentation $W \subseteq V$. Let

$$\hat{\mathfrak{g}} := N \cap \bigcap \{\mathfrak{g}_W \mid W \subseteq V \text{ is a } \mathfrak{g}\text{-subrepresentation}\}.$$

Notice that for any $\mathfrak{g}$-subrepresentation $W \subseteq V$ and $x \in \mathfrak{g}$ it follows from $x(W) \subseteq W$ that als $x_s(W) \subseteq W$ and $x_n(W) \subseteq W$. Because $x_n|_W$ is nilpotent it also follows that $\operatorname{tr} x_n|_W = 0$ and therefore also $\operatorname{tr} x_s|_W = \operatorname{tr} x|_W - \operatorname{tr} x_n|_W = 0$. Hence $x_s, x_n \in \mathfrak{g}_W$.

Combining the above observations it follows that $\hat{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ which contains $\mathfrak{g}$ as an ideal and for which $x_s, x_n \in \hat{\mathfrak{g}}$ for every $x \in \mathfrak{g}$. By Lemma 3.4.5 there exists an ideal $I \subseteq \hat{\mathfrak{g}}$ with $\hat{\mathfrak{g}} = \mathfrak{g} \oplus I$. Let $x \in I$ and $W \subseteq V$ be an irreducible subrepresentation. By construction of $\hat{\mathfrak{g}}$ the action of $\mathfrak{g}$ on W extends to the natural action of $\hat{\mathfrak{g}}$ on W. Because $[\mathfrak{g}, I] = 0$ it follows that $x \in I$ is a homomorphisms of representations of $\mathfrak{g}$. By Schur's Lemma it follows that $x = \lambda \operatorname{id}_W$ for some $\lambda \in k$. Because $\lambda \dim W = \operatorname{tr} x|_W = 0$ this scalar has to be zero. Hence $x = 0$ and therefore $I = 0$. Thus $\hat{\mathfrak{g}} = \mathfrak{g}$. □

Lemma 3.4.2. *Let A be a finite dimensional k-algebra, not necessarily associative nor unital, and $\delta \in \operatorname{Der} A$. For every $\lambda \in k$ let*

$$A_\lambda := \bigcup_{n \in \mathbb{N}} \ker(\delta - \lambda I)^n = \{x \in A \mid (\delta - \lambda I)^m(x) = 0 \text{ for some } m \in \mathbb{N}\}$$

be the generalized eigenspace of δ with respect to λ. Then $A = \bigoplus_{\lambda \in k} A_\lambda$ and

$$A_\lambda A_\mu \subseteq A_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in k.$$

Proof. For this proof abbreviate $1 := I$. That $A = \bigoplus_{\lambda \in k} A_\lambda$ is a standard fact from linear algebra. Let $x \in A_\lambda$ and $y \in A_\mu$. It needs to be shown that $xy \in A_{\lambda+\mu}$, i.e. $(\delta - (\lambda+\mu)I)^m(xy) = 0$ for some $m \in \mathbb{N}$. Because $(\delta - \lambda I)^{m_1}(x) = 0$ and $(\delta - \mu I)^{m_2}(y) = 0$ for some $m_1, m_2 \in \mathbb{N}$ this follows from the formula

$$(\delta - (\lambda+\mu)I)^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (\delta - \lambda I)^i(x)(\delta - \mu I)^{n-i}(y) \quad \text{for every } n \in \mathbb{N},$$

which can be shown by induction over $n \in \mathbb{N}$: For $n = 0$ the statement holds. Suppose it holds for some $n \in \mathbb{N}$. Then

$$(\delta - (\lambda+\mu)I)^{n+1}(xy) = (\delta - (\lambda+\mu)I)((\delta - (\lambda+\mu)I)^n(xy))$$

$$
\begin{aligned}
&= (\delta - (\lambda + \mu)I)\left(\sum_{i=0}^{n}\binom{n}{i}(\delta - \lambda I)^i(x)(\delta - \mu I)^{n-i}(y)\right) \\
&= \sum_{i=0}^{n}\binom{n}{i}\delta\left((\delta - \lambda I)^i(x)\right)(\delta - \mu I)^{n-i}(y) + \sum_{i=0}^{n}\binom{n}{i}(\delta - \lambda I)^i(x)\delta\left((\delta - \mu I)^{n-i}(y)\right) \\
&\quad - \sum_{i=0}^{n}\binom{n}{i}\lambda(\delta - \lambda I)^i(x)(\delta - \mu I)^{n-i}(y) - \sum_{i=0}^{n}\binom{n}{i}(\delta - \lambda I)^i(x)\mu(\delta - \mu I)^{n-i}(y) \\
&= \sum_{i=0}^{n}\binom{n}{i}(\delta - \lambda I)^{i+1}(x)(\delta - \mu I)^{n-i}(y) + \sum_{i=0}^{n}\binom{n}{i}(\delta - \lambda I)^i(x)(\delta - \mu I)^{n+1-i}(y) \\
&= \sum_{i=1}^{n+1}\binom{n}{i-1}(\delta - \lambda I)^i(x)(\delta - \mu I)^{n+1-i}(y) + \sum_{i=0}^{n}\binom{n}{i}(\delta - \lambda I)^i(x)(\delta - \mu I)^{n+1-i}(y) \\
&= \sum_{i=1}^{n}\left(\binom{n}{i-1} + \binom{n}{i}\right)(\delta - \lambda I)^i(x)(\delta - \mu I)^{n+1-i}(y) \\
&\quad + (\delta - \mu I)^{n+1}(y) + (\delta - \lambda I)^{n+1}(x) \\
&= \sum_{i=1}^{n}\binom{n+1}{i}(\delta - \lambda I)^i(x)(\delta - \mu I)^{n+1-i}(y) + (\delta - \mu I)^{n+1}(y) + (\delta - \lambda I)^{n+1}(x) \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}(\delta - \lambda I)^i(x)(\delta - \mu I)^{n+1-i}(y)
\end{aligned}
$$
□

Remark 3.4.3. Notice that the formula in the proof of Lemma 3.4.2 is just a generalization of the binomial theorem, which follows by setting $\delta = 0$.

Lemma 3.4.4. *Let A be a finite dimensional k-algebra, not necessarily associative nor unital, e.g. a Lie algebra. Then $\operatorname{Der}(A)$ contains the semisimple and nilpotent parts (in $\operatorname{End}_k(A)$) of all its elements.*

Proof. Let $\delta \in \operatorname{Der}(A)$ and for every $\lambda \in k$ let A_λ be the generalized eigenspace with respect to λ. To show that $\operatorname{Der}(A)$ contains the semisimple and nilpotent part of δ is sufficies to do so for the semisimple part, which will be denoted by σ. As seen in the proof of Theorem 2.2.17 (the concrete Jordan decomposition) σ acts on A_λ by multiplication with λ for every $\lambda \in k$. If $x \in A_\lambda$ and $y \in A_\mu$ then $xy \in A_{\lambda+\mu}$ and thus

$$\sigma(xy) = (\lambda + \mu)(xy) = (\lambda x)y + x(\mu y) = \sigma(x)y + x\sigma(y).$$

Because $A = \bigoplus_{\lambda \in k} A_\lambda$ it follows that σ is a derivation of A. □

Lemma 3.4.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $I \trianglelefteq \mathfrak{g}$ a semisimple ideal. Then there exists an ideal $J \trianglelefteq \mathfrak{g}$ with $\mathfrak{g} = I \oplus J$.*

Proof. Because $I \trianglelefteq \mathfrak{g}$ is an ideal it follows that $\kappa_I = \kappa_{\mathfrak{g}}|_{I \times I}$. Because I is semisimple it also follows that κ_I is non-degenerate. It follows for the linear map

$$\varphi \colon \mathfrak{g} \to I^*, \quad x \mapsto \kappa(x, \cdot)$$

that $\varphi|_I : I \to I^*$ is an isomorphism. Hence $I \cap \ker \varphi = 0$ with

$$\dim I = \dim I^* = \dim \mathfrak{g} - \dim \ker \varphi.$$

Let $J \coloneqq I^\perp$ be the orthogonal complement of I with respect to the Killing form. From Lemma 2.2.6 it is known that J is an ideal in $\mathfrak{g}$. Because $J = \ker \varphi$ it follows from the previous observations that $I \cap J = 0$ and $\dim \mathfrak{g} = \dim I + \dim J$ and thus $\mathfrak{g} = I \oplus J$. ☐

Lemma 3.4.6. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then every derivation of $\mathfrak{g}$ is inner.*

Proof. As seen in Lemma 1.1.38 and its proof the inner derivations form an ideal $I \trianglelefteq \operatorname{Der}(\mathfrak{g})$ with $[\delta, \operatorname{ad}(x)] = \operatorname{ad}(\delta(x))$ for every $\delta \in \operatorname{Der}(\mathfrak{g})$ and $x \in \mathfrak{g}$. By Lemma 3.4.5 there exists an ideal $J \trianglelefteq \operatorname{Der}(\mathfrak{g})$ with $\operatorname{Der}(\mathfrak{g}) = I \oplus J$. Let $\delta \in J$. Then for any $x \in \mathfrak{g}$

$$\operatorname{ad}(\delta(x)) = [\delta, \operatorname{ad}(x)] \in J \cap I = 0,$$

and thus $\operatorname{ad}(\delta(x)) = 0$. Because ad is injective it follows that $\delta(x) = 0$ for every $x \in \mathfrak{g}$ and thus $\delta = 0$. Therefore $J = 0$ and thus $\operatorname{Der}(A) = I$. ☐

Theorem 3.4.7 (Abstract Jordan decomposition)**.** *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then for any $x \in \mathfrak{g}$ there exist unique elements $x_s, x_n \in \mathfrak{g}$ such that $\operatorname{ad}(x) = \operatorname{ad}(x_s) + \operatorname{ad}(x_n)$ is the concrete Jordan decomposition of $\operatorname{ad}(x)$ with $\operatorname{ad}(x_s) = \operatorname{ad}(x)_s$ and $\operatorname{ad}(x_n) = \operatorname{ad}(x)_n$.*

Then $x = x_s + x_n$ and an arbitrary element of $\mathfrak{g}$ commutes with x if and only if it commutes both with x_s and x_n. In particular x, x_s and x_n are pairwise commuting.

Proof. By Lemma 3.4.6 the adjoint representation $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{ad}(\mathfrak{g}) = \operatorname{Der}(\mathfrak{g})$ is an isomorphism of Lie algebras. By Lemma 3.4.4 $\operatorname{Der}(\mathfrak{g})$ contains the semisimple and nilpotent parts of all its elements. By setting $x_s :- \operatorname{ad}^{-1}(\operatorname{ad}(x)_s)$ and $x_n \coloneqq \operatorname{ad}^{-1}(\operatorname{ad}(x)_n)$ the theorem follows from the properties of the concrete Jordan decomposition as stated in Theorem 2.2.17. ☐

Definition 3.4.8. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and $x \in \mathfrak{g}$. Then the decomposition $x = x_s + x_n$ as is Theorem 3.4.7 is the *abstract Jordan decomposition* of x where x_s is the *abstract semisimple part* and x_n is the *abstract nilpotent part* of x.

Remark 3.4.9. By Definition 3.4.8 an element of a finite dimensional semisimple Lie algebra is semisimple if and only if it is ad-semisimple and nilpotent if and only if it is ad-nilpotent.

Proposition 3.4.10. *Let $\mathfrak{g}$ be a finite dimensional semisimple linear Lie algebra. Then the abstract and concrete Jordan decomposition coincide.*

Proof. Let $x \in \mathfrak{g}$, $x = xc_s + x_n$ the abstract Jordan decomposition of x and $x = y_s + y_n$ the concrete Jordan decomposition of x. From Lemma 3.4.1 it follows that $y_s, y_n \in \mathfrak{g}$ and by Lemma 2.2.21 $\operatorname{ad}(x) = \operatorname{ad}(y_s) + \operatorname{ad}(y_n)$ is the concrete Jordan decomposition of $\operatorname{ad}(x)$ with $\operatorname{ad}(y_s) = \operatorname{ad}(x)_s$ and $\operatorname{ad}(y_n) = \operatorname{ad}(x)_n$. Hence y_s is the abstract semisimple part of x and y_n is the abstract nilpotent part of x. ☐

Remark 3.4.11. By Proposition 3.4.10 we need not to distinguish between the concrete and abstract Jordan decomposition when working with a finite dimensional semisimple linear Lie algebra, which is why we will stop doing so.

Lemma 3.4.12. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algbra, $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$ and $x \in \mathfrak{g}$. If $x = x_s + x_n$ is the Jordan decomposition of x then $\rho(x) = \rho(x_s) + \rho(x_n)$ is the Jordan decomposition of $\rho(x)$ with $\rho(x_s) = \rho(x)_s$ and $\rho(x_n) = \rho(x)_n$.*

Proof. Let $x \in \mathfrak{g}$ be semisimple (resp. nilpotent). Then x is $\mathrm{ad}_{\mathfrak{g}}$-semisimple (resp. $\mathrm{ad}_{\mathfrak{g}}$-nilpotent). It follows that $\rho(x)$ is $\mathrm{ad}_{\rho(\mathfrak{g})}$-semisimple (resp. $\mathrm{ad}_{\rho(\mathfrak{g})}$-nilpotent) and therefore semisimple (resp. nilpotent). □

Corollary 3.4.13 (Functoriality of the Jordan decomposition)**.** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be finite dimensional semisimple Lie algebras and $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ a homomorphism of Lie algebras. Then ϕ preserves the Jordan decomposition, i.e. $\phi(x_s) = \phi(x)_s$ and $\phi(x_n) = \phi(x)_n$ for every $x \in \mathfrak{g}_1$.*

Proof. By Lemma 3.4.12 the homomorphism of Lie algebras $\mathrm{ad}_{\mathfrak{g}_2} \circ \phi\colon \mathfrak{g}_1 \to \mathfrak{gl}(\mathfrak{g}_2)$ preserves the Jordan decomposition. The homomorphism $\mathrm{ad}_{\mathfrak{g}_2}^{-1}\colon \mathrm{ad}(\mathfrak{g}_2) \to \mathfrak{g}_2$ preserves the Jordan decomposition by the definition of the abstract Jordan decomposition. Hence $\phi = \mathrm{ad}_{\mathfrak{g}_2}^{-1} \circ \mathrm{ad}_{\mathfrak{g}_2} \circ \phi$ preserves the Jordan decomposition as well. □

3.5. Cartan subalgebras

Throughout this section $\mathfrak{g}$ denotes the finite dimensional semisimple Lie algebra.

3.5.1. Definition and root space decomposition

Definition 3.5.1. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *toral* if it consists of semisimple elements.

Lemma 3.5.2. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a toral subalgebra. Then $\mathfrak{h}$ is abelian.*

Proof. Let $x \in \mathfrak{h}$. Because x is $\mathrm{ad}_{\mathfrak{g}}$-semisimple and $\mathfrak{h}$ is $\mathrm{ad}_{\mathfrak{g}}(x)$-invariant it follows that $\mathrm{ad}_{\mathfrak{h}}(x) = \mathrm{ad}_{\mathfrak{g}}(x)|_{\mathfrak{h}}$ is also semisimple. It is enough to show that all eigenvalues of $\mathrm{ad}_{\mathfrak{h}}(x)$ are zero.

Let $y \in \mathfrak{h}$ be an eigenvector of $\mathrm{ad}_{\mathfrak{h}}(x)$ with eigenvalue μ (in particular $y \neq 0$). In the same way as for $\mathrm{ad}_{\mathfrak{h}}(x)$ it follows that $\mathrm{ad}_{\mathfrak{h}}(y)$ is semisimple. Fer every $\lambda \in k$ let $\mathfrak{h}_\lambda$ denote the eigenpsace of $\mathrm{ad}_{\mathfrak{h}}(y)$ with respect to the eigenvalue λ.

On the one hand $[y, x] \in \mathfrak{h}_0$ because

$$\mathrm{ad}_{\mathfrak{h}}(y)([y, x]) = [y, [y, x]] = [y, -\mu y] = 0.$$

On the other hand

$$[y, x] = \mathrm{ad}_{\mathfrak{h}}(y)(x) \in \mathrm{ad}_{\mathfrak{h}}(y)(\mathfrak{h}) = \mathrm{ad}_{\mathfrak{h}}(y)\left(\bigoplus_{\lambda \in k} \mathfrak{h}_\lambda\right) = \bigoplus_{\lambda \neq 0} \mathfrak{h}_\lambda.$$

By the directness of the sum $\mathfrak{h} = \bigoplus_{\lambda \in k} \mathfrak{h}_\lambda$ it follows that $0 = [y, x] = -\mu y$. Because $y \neq 0$ it follows that $\mu = 0$. □

Definition 3.5.3. A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal toral subalgebra.

Remark 3.5.4. The toral subalgebra $0 \subseteq \mathfrak{g}$ is contained in a toral subalgebra (of $\mathfrak{g}$) of maximal dimensional, which is then a Cartan subalgebra. Therefore $\mathfrak{g}$ contains a Cartan subalgebra.

Also notice that if $\mathfrak{g} \neq 0$ then any Cartan subalgebra of $\mathfrak{g}$ is non-zero. Too see this first notice thet $\mathfrak{g}$ then contains a non-zero semisimple element $x \in \mathfrak{g}$, because otherwise $\mathfrak{g}$ would be nilpotent by Engel's Theorem. Then the one-dimensional linear subspace kx is a toral subalgebra properly containing 0.

Definition 3.5.5. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For any $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha \coloneqq \{y \in \mathfrak{g} \mid [x, y] = \alpha(x)y \text{ for every } x \in \mathfrak{h}\}$$

be the *root space* of $\mathfrak{g}$ with respect to α. Then $\Phi(\mathfrak{g}, \mathfrak{h}) \coloneqq \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is the set of *roots* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Remark 3.5.6. Notice that $\mathfrak{g}_0 = \{y \in \mathfrak{g} \mid [x, y] = 0 \text{ for every } x \in \mathfrak{h}\} = Z_{\mathfrak{g}}(\mathfrak{h})$.

Lemma 3.5.7. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with roots $\Phi \coloneqq \Phi(\mathfrak{g}, \mathfrak{h})$. Then $\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.*

Proof. Because $\mathfrak{h}$ is abelian the same goes for $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{gl}(V)$. Therefore $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{h})$ consists of semisimple pairwise commuting endomorphisms. Hence $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{h})$ is simultaneously diagonalizable, which is why

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$ □

Lemma 3.5.8. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{h}^*$.*

Proof. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then for every $h \in \mathfrak{h}$

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y].$$ □

Lemma 3.5.9. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and $\alpha, \beta \in \mathfrak{h}^*$. If $\alpha \neq -\beta$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form.*

Proof. Because $\alpha \neq -\beta$ it follows that there exists $h \in \mathfrak{h}$ with $(\alpha + \beta)(h) \neq 0$. For every $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ it then follows that

$$\alpha(h)\kappa(x, y) = \kappa([h, x], y) = -\kappa([x, h], y) = -\kappa(x, [h, y]) = -\beta(h)\kappa(x, y)$$

and therefore $(\alpha + \beta)(h)\kappa(x, y) = 0$. Because $(\alpha + \beta)(h) \neq 0$ it follows that $\kappa(x, y) = 0$ for every $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. □

Corollary 3.5.10. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then $\kappa_{\mathfrak{g}}|_{Z_{\mathfrak{g}}(\mathfrak{h}) \times Z_{\mathfrak{g}}(\mathfrak{h})}$ is non-degenerate.*

Proof. Because $\mathfrak{g}$ is semisimple $\kappa_{\mathfrak{g}}$ is non-degenerate. Because $Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$ is orthogonal to $\mathfrak{g}_\alpha$ for every $\alpha \in \Phi$ with $\alpha \neq 0$ the statement follows from $\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. □

Proposition 3.5.11. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, i.e. $\mathfrak{h}$ is self-centralizing.*

Proof. Throughout this proof abbreviate $\mathfrak{c} := Z_{\mathfrak{g}}(\mathfrak{h})$, $\operatorname{ad} := \operatorname{ad}_{\mathfrak{g}}$ and $\kappa := \kappa_{\mathfrak{g}}$.

Claim. *Let $x \in \mathfrak{g}$ be nilpotent and $y \in \mathfrak{g}$ commuting with x. Then $\kappa(x, y) = 0$.*

Proof. Because x and y commute so do $\operatorname{ad}(x)$ and $\operatorname{ad}(y)$. Because $\operatorname{ad}(x)$ is nilpotent it follows that $\operatorname{ad}(x)\operatorname{ad}(y)$ is also nilpotent. Therefore $\kappa(x, y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$. □

Start by noticing that $\mathfrak{c}$ contains the semisimple and nilpotent parts of all its elements: If $x \in \mathfrak{c}$ then $y \in \mathfrak{g}$ commutes with x if and only if it commutes with both y_s and y_n. Therefore $y \in \mathfrak{c}$ if and only if $y_s, y_n \in \mathfrak{c}$.

Let $s \in \mathfrak{c}$ be semisimple. Because s is semisimple and commutes with $\mathfrak{h}$ it then follows that $\mathfrak{h} + ks$ is a Lie subalgebra of $\mathfrak{g}$ consisting of semisimple elements. By the maximality of $\mathfrak{h}$ it follows that $s \in \mathfrak{h}$.

Let $x \in \mathfrak{c}$ and let $x = x_s + x_n$ be the Jordan decomposition of x. It was already shown above that $x_s \in \mathfrak{h}$, so $\operatorname{ad}_{\mathfrak{c}} x_s = 0$. Hence $\operatorname{ad}_{\mathfrak{c}} x = \operatorname{ad}_{\mathfrak{c}} x_n = \operatorname{ad}_{\mathfrak{g}} x_n|_{\mathfrak{c}}$ is nilpotent. By Engel's Theorem $\mathfrak{c}$ is nilpotent.

Notice that $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. To see this let $x \in \mathfrak{h}$ with $\kappa(x, \mathfrak{h}) = 0$. It needs to be shown that $x = 0$, and for this it sufficies to show that $\kappa(x, \mathfrak{c}) = 0$ by Corollary 3.5.10. Because $\mathfrak{c}$ contains the semisimple and nilpotent parts of all its elements is sufffices to show that $\kappa(x, s) = \kappa(x, n) = 0$ for every semisimple $s \in \mathfrak{c}$ and nilpotent $n \in \mathfrak{c}$. It was already shown that $s \in \mathfrak{h}$ so $\kappa(x, s) = 0$ by assumption. That $\kappa(x, n) = 0$ follows from the claim above.

It follows that $\mathfrak{h} \cap [\mathfrak{c}, \mathfrak{c}] = 0$ because $[\mathfrak{h}, \mathfrak{c}] = 0$ and thus $\kappa(\mathfrak{h}, [\mathfrak{c}, \mathfrak{c}]) = \kappa([\mathfrak{h}, \mathfrak{c}], \mathfrak{c}) = 0$. It further follows that $\mathfrak{c}$ is abelian. Otherwise $\mathfrak{c}$ is nilpotent with $[\mathfrak{c}, \mathfrak{c}] \neq 0$. Then $Z(\mathfrak{c}) \cap [\mathfrak{c}, \mathfrak{c}] \neq 0$ by Corollary 2.1.15. Let $x \in Z(\mathfrak{c}) \cap [\mathfrak{c}, \mathfrak{c}]$ be non-zero. Notice that x cannot be semisimple because $\mathfrak{h} \cap [\mathfrak{c}, \mathfrak{c}] = 0$. So x_n is nonzero and it was already shown that $x_n \in \mathfrak{c}$. Because $x \in Z(\mathfrak{c})$ it follows that $x_n \in Z(\mathfrak{c})$. From the claim above it follows that $\kappa(x_n, \mathfrak{c}) = 0$. Because $\kappa_{\mathfrak{c} \times \mathfrak{c}}$ is non-degenerate it follows that $x_n = 0$, contradicting that x_n is non-zero.

Suppose now that $\mathfrak{h} \subsetneq \mathfrak{c}$. Let $x \in \mathfrak{c}$ with $\mathfrak{h} \neq 0$. Because $x_s, x_n \in \mathfrak{c}$ with $x_s \in \mathfrak{h}$ it can be assumed w.l.o.g. that x is nilpotent by replacing it with x_n. Then $\kappa(x_n, \mathfrak{c}) = 0$ by the above claim, contradicting $\kappa_{\mathfrak{c} \times \mathfrak{c}}$ being non-degenerate. □

Corollary 3.5.12. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then the restriction $\kappa_{\mathfrak{g}}|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha$.*

Corollary 3.5.13. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. Then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$, i.e. there exists $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ with $\kappa(x, y) \neq 0$.*

Proof. Because $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ it follows that $\mathfrak{g}_\alpha \neq 0$. Because with respect to the Killing form $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ for $\beta \neq -\alpha$ and $\mathfrak{h} = \mathfrak{g}_0$ by Lemma 3.5.9 and $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate is follows that $0 \neq \kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = \kappa(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha})$. □

Remark 3.5.14. The proof of Proposition 3.5.11 is taken from [**Humphreys**].

Definition 3.5.15. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \quad \text{with} \quad \Phi := \Phi(\mathfrak{g}, \mathfrak{h})$$

is the *root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

3.5.2. Properties of the root space decomposition

Troughout this subsection let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and $\Phi := \Phi(\mathfrak{g}, \mathfrak{h})$ the associated roots.

Definition 3.5.16. Becaus $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate the map

$$\mathfrak{h} \to \mathfrak{h}^*, \quad h \mapsto \kappa(h, \cdot)$$

is an isomorphism of vector spaces. For every $\phi \in \mathfrak{h}^*$ let $t_\phi \in \mathfrak{h}$ be the unique element with $\kappa(t_\phi, \cdot) = \phi$.

Proposition 3.5.17. *1. Φ generates $\mathfrak{h}^*$ as a vector space.*

2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

3. $[x, y] = \kappa(x, y)t_\alpha$ for every $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$.

4. Lef $\alpha \in \Phi$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional and has t_α as basis.

5. If $\alpha \in \Phi$ then $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

Proof. 1. Suppose Φ does not generates $\mathfrak{h}^*$ as a vector space. Then there exists some $\phi \in \mathfrak{h}^*$ with $\phi \notin \operatorname{span}_k \Phi$. Then there exists some $y \in \mathfrak{h}^{**}$ with $y|_{\operatorname{span}_k \Phi} = 0$ but $y(\phi) \neq 0$. By the natural isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^{**}, x \mapsto (\psi \mapsto \psi(x))$ there exists some $x \in \mathfrak{h}$ with $y(\psi) = \psi(x)$ for every $\psi \in \mathfrak{h}^*$. In particular $\alpha(x) = 0$ for every $\alpha \in \mathfrak{h}^*$ but $\phi(x) \neq 0$ and thus $x \neq 0$. Using the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ it follows that $x \in Z(\mathfrak{g})$. Because $Z(\mathfrak{g}) = 0$ this contradicts x being non-zero.

2. Let $\alpha \in \Phi$. Then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$ by Corollary 3.5.13, from which it follows that $\mathfrak{g}_{-\alpha} \neq 0$ and therefore $-\alpha \in \Phi$.

3. Let $h \in \mathfrak{h}$. Then

$$\begin{aligned}\kappa(h, [x, y]) &= \kappa([h, x], y) = \alpha(h)\kappa(x, y) \\ &= \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h) = \kappa(h, \kappa(x, y)t_\alpha).\end{aligned}$$

Becasue $\kappa_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate it follows that $[x, y] = \kappa(x, y)t_\alpha$.

4. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ with $\kappa(x,y) \neq 0$ (such elements exist by Corollary 3.5.13). Then $[x,y] = \kappa(x,y)t_\alpha \neq 0$ and therefore $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \kappa(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha})t_\alpha = kt_\alpha$.

5. Suppose that there exist some $\alpha \in \Phi$ with $\kappa(t_\alpha, t_\alpha) = 0$. Because $\alpha = \kappa(t_\alpha, \cdot)$ it follows that $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = 0$. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ with $\kappa(x,y) \neq 0$. By rescaling it can be additionally assumed that $\kappa(x,y) = 1$ and thus $[x,y] = \kappa(x,y)t_\alpha = t_\alpha$. Then $L \coloneqq \operatorname{span}_k(x, t_\alpha, y)$ is a three-dimensional solvable Lie subalgebra of $\mathfrak{g}$ because $[t_\alpha, x] = \alpha(t_\alpha)x = 0$ and $[t_\alpha, y] = -\alpha(t_\alpha)y = 0$.

   It follows that $\operatorname{ad}(L)$ is a three-dimensional solvable Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ and thus $[\operatorname{ad}(L), \operatorname{ad}(L)] = \operatorname{ad}([L,L]) = k \operatorname{ad}(t_\alpha)$ consists of nilpotent endomorphisms of $\mathfrak{g}$. (To see this notice that by Lie's theorem there exists a basis of $\mathfrak{g}$ with respect to which $\operatorname{ad}(L)$ is represented by upper triangular matrices. Then with respect to this basis $[\operatorname{ad}(L), \operatorname{ad}(L)]$ is represented by strictly upper triangular matrices.) In particular $\operatorname{ad}(t_\alpha)$ is nilpotent. On the other hand t_α is semisimple (because $t_\alpha \in \mathfrak{h}$ and $\mathfrak{h}$ consists of semisimple elements by the definition of a toral subalgebra) and thus $\operatorname{ad}(t_\alpha)$ is semisimple. Because $\operatorname{ad}(t_\alpha)$ is both nilpotent is semisimple it follows that $\operatorname{ad}(t_\alpha) = 0$ and thus $t_\alpha = 0$, contradicting $\kappa(t_\alpha, \cdot) = \alpha \neq 0$. □

Definition 3.5.18. Let $\alpha \in \Phi$ be a root. Then the associated *coroot* is the unique element $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(\alpha^\vee) = 2$.

Remark 3.5.19. Let $\alpha \in \Phi$. The existence and uniqueness of $\alpha^\vee$ follows from the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional with $\alpha([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = k\alpha(t_\alpha) \neq 0$. Notice that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = k\alpha^\vee$ and that $(-\alpha)^\vee = -\alpha^\vee$. Also notice that $\alpha^\vee = 2t_\alpha/\kappa(t_\alpha, t_\alpha)$ because

$$\alpha\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right) = \frac{2\alpha(t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \frac{2\kappa(t_\alpha, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = 2.$$

Remark 3.5.20. Let $\alpha \in \Phi$. Let $S^\alpha \coloneqq \mathfrak{g}_\alpha \oplus k\alpha^\vee \oplus \mathfrak{g}_{-\alpha}$. Because $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = k\alpha^\vee$ there exist $\hat{e} \in \mathfrak{g}_\alpha$ and $\hat{f} \in \mathfrak{g}_{-\alpha}$ with $[\hat{e}, \hat{f}] = \alpha^\vee$. Because $\alpha^\vee \neq 0$ it follows that $\hat{e}, \hat{f} \neq 0$. Also notice that $[\alpha^\vee, \hat{e}] = \alpha(\alpha^\vee)\hat{e} = 2\hat{e}$ and similary $[\alpha^\vee, \hat{f}] = -2\hat{f}$. It follows that S^α is a Lie subalgebra of $\mathfrak{g}$ and

$$\phi \colon \mathfrak{sl}_2(k) \to S, \quad e \mapsto \hat{e}, \quad h \mapsto \alpha^\vee, \quad f \mapsto \hat{f}$$

an isomorphism of Lie algebras. Therefore $\mathfrak{g}$ becomes a representation of $\mathfrak{sl}_2(k)$ via $x.y = [\phi(x), y]$ for every $x \in \mathfrak{sl}_2(k)$ and $y \in \mathfrak{g}$.

Definition 3.5.21. Let $\alpha \in \Phi$. Then $S^\alpha \subseteq \mathfrak{g}$ is the Lie subalgebra construtcted in Remark 3.5.20.

Proposition 3.5.22. *Let $\alpha \in \Phi$ be a root.*

1. *The root space $\mathfrak{g}_\alpha$ is one-dimensional.*

2. *The only multiples of α which are roots are α and $-\alpha$, i.e. $k\alpha \cap \Phi = \{\alpha, -\alpha\}$.*

Proof. Let S^α acts on $\mathfrak{g}$ as in Remark 3.5.20. Notice that

$$L := k\alpha^\vee \oplus \bigoplus_{\substack{c \in k \\ c\alpha \in \Phi}} \mathfrak{g}_{c\alpha}$$

is a subrepresentation of S^α.

By Weyl‘s theorem L is completely reducible. Notice that $h.\alpha^\vee = [\alpha^\vee, \alpha^\vee] = 0$ and

$$h.x = [\alpha^\vee, x] = c\alpha(\alpha^\vee)(x) = 2cx \quad \text{for every } x \in \mathfrak{g}_{c\alpha}.$$

Because all weights of L are integral it follows that $2c \in \mathbb{Z}$ for every $c \in k$ with $c\alpha \in \Phi$ and thus $c \in \frac{1}{2}\mathbb{Z}$. From the classification of the finite dimensional representations of $\mathfrak{sl}_2(k)$ it follows that L is the direct sum of irreducible submodules, of which there are two kinds: Those with even weights and those with odd weights; the number of summands of the first kind is given by $\dim L_0$ and the number of summands of the second kind is given by $\dim L_1$.

Now $\dim L_0 = \dim k\alpha^\vee = 1$, therefore a decomposition of L into irreducible subrepresentations has exactly one summand with even weights. As $S \subseteq L$ is a subrepresentation with even even weights, namely $S^\alpha_{-2} = k\hat{f}$, $S^\alpha_0 = k\alpha^\vee$ and $S^\alpha_2 = k\hat{e}$, we have found this summand. In particular 2α is not a root, resulting in the following:

Claim. *Let $\beta \in \Phi$ be a root. Then 2β is not it root, i.e. twice a root is never a root. Equivalently half a root is never a root.*

From this claim it follows that $\alpha/2$ is not root, so $L_1 = \mathfrak{g}_{\alpha/2} = 0$. Hence L contains no summand with odd weights and therefore already $L = S^\alpha$. As a direct consequence $\dim \mathfrak{g}_\alpha = \dim L_2 = \dim S^\alpha_2 = 1$ and $k\alpha \cap \Phi = \{c \in k \setminus \{0\} \mid L_{2c} \neq 0\} = \{1, -1\}$. ☐

Definition 3.5.23. For every $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$ set $\langle h, \lambda \rangle := \langle \lambda, h \rangle := \lambda(h)$.

Proposition 3.5.24. *Let $\alpha, \beta \in \Phi$. Then $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ and $\alpha - \langle \alpha, \beta^\vee \rangle \beta \in \Phi$.*

Proof. If α and β are linearly dependent then $\beta = \pm\alpha$. Then $\beta^\vee = \pm\alpha^\vee$ and therefore $\langle \alpha, \beta^\vee \rangle = \pm \langle \alpha, \alpha^\vee \rangle = \pm 2 \in \mathbb{Z}$ and

$$\alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha - 2\alpha = -\alpha \in \Phi.$$

Suppose that α and β are linearly independent. Let S^β act on $\mathfrak{g}$ as in Remark 3.5.20 and notice that $L := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\alpha + i\beta}$ is a subrepresentation. For $x \in \mathfrak{g}_{\alpha+i\beta}$ it follows that

$$h.x = [\beta^\vee, x] = (\alpha + i\beta)(\beta^\vee)x = (\langle \alpha, \beta^\vee \rangle + 2i)x,$$

so $\mathfrak{g}_{\alpha+i\beta} = L_{\langle \alpha, \beta^\vee \rangle + 2i}$ for every $i \in \mathbb{Z}$. Because $L_{\langle \alpha, \beta^\vee \rangle} = \mathfrak{g}_\alpha \neq 0$ it follows that $\langle \alpha, \beta^\vee \rangle$ is a weight of L and therefore integral by $\mathfrak{sl}_2$-theory. From $\mathfrak{sl}_2$-theory it also follows that $-\langle \alpha, \beta^\vee \rangle$ is also a weight of L. Therefore there exists some $i \in \mathbb{Z}$ with $\langle \alpha, \beta^\vee \rangle + 2i = -\langle \alpha, \beta^\vee \rangle$, namely $i = -\langle \alpha, \beta^\vee \rangle$. Then

$$\mathfrak{g}_{\alpha - \langle \alpha, \beta^\vee \rangle \beta} = \mathfrak{g}_{\alpha + i\beta} = L_{\langle \alpha, \beta^\vee \rangle + 2i} = L_{\langle \alpha, \beta^\vee \rangle - 2\langle \alpha, \beta^\vee \rangle} = L_{-\langle \alpha, \beta^\vee \rangle} \neq 0,$$

which shows that $\alpha - \langle \alpha, \beta^\vee \rangle \beta \in \Phi$. ☐

Corollary 3.5.25. *Let $\alpha, \beta \in \Phi$. Then*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is already known that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. If $\alpha + \beta \notin \Phi$ then $\mathfrak{g}_{\alpha+\beta} = 0$ and thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. Suppose now that $\alpha + \beta \in \Phi$. Then α and β are linearly independent (because otherwise $\beta = \pm\alpha$ and therefore $\alpha + \beta \in \{-2\alpha, 0, 2\alpha\}$, none of which is a root). Let $\mathfrak{g}^\alpha$ act on $\mathfrak{g}$ as in Remark 3.5.20 and consider the subrepresentation $L \coloneqq \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$.

As seen in the proof of Proposition 3.5.24 it follows that $\mathfrak{g}_{\beta+i\alpha} = L_{\langle\beta,\alpha^\vee\rangle+2i}$ for every $i \in \mathbb{Z}$. In particular every nonzero weight space of L is one-dimensional and all weights have the same parity. Hence L is an irreducible representation of $\mathfrak{g}^\alpha$.

Let $\hat{e} \in \mathfrak{g}_\alpha \subseteq S^\alpha$ be the element corresponding to $e \in \mathfrak{sl}_2(k)$ under the isomorphism $\mathfrak{sl}_2(k) \cong S^\alpha$ which was used to construct the action of $\mathfrak{sl}_2(k)$ on $\mathfrak{g}$. Because $\beta, \alpha+\beta \in \Phi$ are both roots it follows that $L_{\langle\beta,\alpha^\vee\rangle} = \mathfrak{g}_\beta$ and $L_{\langle\beta,\alpha^\vee\rangle+2} = \mathfrak{g}_{\alpha+\beta}$ are both nonzero. Therefore it follows from $\mathfrak{sl}_2$-theory that $e.L_{\langle\beta,\alpha^\vee\rangle} = L_{\langle\beta,\alpha^\vee\rangle+2}$. But this is the same as saying that $[\hat{e}, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ and thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. □

4. Root Systems

4.1. Abstract root systems

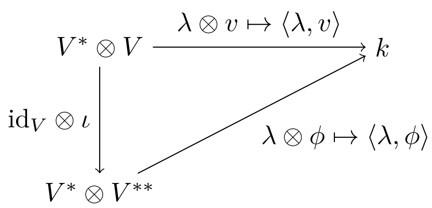
For this section let k be an arbitrary field with $\operatorname{char} k = 0$.

Definition 4.1.1. Let V be a k-vector space. For every $v \in V$ and $\lambda \in V^*$ set

$$\langle v, \lambda \rangle := \langle \lambda, v \rangle := 2\lambda(v).$$

Remark 4.1.2. Notice that the bilinear form $V^* \times V \to k, (\lambda, v) \mapsto \langle \lambda, v \rangle = \lambda(v)$ is non-degenerate: If $v \in V$ with $\lambda(v) = 0$ for every $\lambda \in V^*$ then $v = 0$ and if $\lambda \in V^*$ with $\lambda(v) = 0$ for every $v \in V$ then $\lambda = 0$.

Remark 4.1.3. Notice that the notation of Definition 4.1.1 is compatible with the natural monomorphism $\iota\colon V \to V^{**}, v \mapsto (\lambda \mapsto \lambda(v))$ in the sense that the following diagram commutes:



If V is finite dimensional then ι is already an isomorphism and identifies V with V^{**}. Then $\langle \phi, v \rangle$ can be understood as $\phi(v)$ or $\iota(v)(\phi)$ and the above shows that both are actually the same.

Definition 4.1.4. Let V be a finite dimensional k-vector space. A subset $R \subseteq V$ is called an *(abstract) root system (in V)* if the following hold:

1. R is finite, $0 \notin R$ and R generates V as a k-vector space.
2. For every $\alpha \in R$ exists some $\alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 2$ and $S_{\alpha,\alpha^\vee}(R) \subseteq R$ for the linear map

$$S_{\alpha,\alpha^\vee}\colon V \to V, \quad \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle\, \alpha.$$

3. $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

The root system R is *reduced* if $k\alpha \cap R = \{-\alpha, \alpha\}$ for every $\alpha \in R$, i.e. if the only multiples of α which are also roots are α and $-\alpha$. The *rank* of the root system R is the dimension of V.

Remark 4.1.5. Notice that if V is a finite dimensional vector space and $R \subseteq V$ a root system with $\alpha \in R$ then also

$$-\alpha = \alpha - 2\alpha = \alpha - \langle \alpha, \alpha^\vee \rangle \alpha = S_{\alpha,\alpha^\vee}(\alpha) \in R.$$

Lemma 4.1.6. *Let V be a finite dimensional k-vector space and $R \subseteq V$ a root system. Then the element $\alpha^\vee \in V$ with $\alpha^\vee(\alpha) = 2$ is unique.*

Proof. Let $\alpha^\vee, \tilde{\alpha}^\vee \in V^*$ with $\alpha^\vee(\alpha) = \tilde{\alpha}^\vee(\alpha) = 2$ and $s(R) \subseteq R$ as well as $t(R) \subseteq R$ for $s := S_{\alpha,\alpha^\vee}$ and $t := S_{\alpha,\tilde{\alpha}^\vee}$. Notice that $s(\alpha) = t(\alpha) = -\alpha$ as already seen in Remark 4.1.5.

By induction on n it follows that

$$(s \circ t)^n(x) = x - n \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle \alpha \quad \text{for every } x \in V \text{ and } n \geq 1. \tag{1}$$

For $n = 1$ this holds because

$$\begin{aligned} s(t(x)) &= s(x - \langle x, \tilde{\alpha}^\vee \rangle \alpha) = s(x) - \langle x, \tilde{\alpha}^\vee \rangle s(\alpha) \\ &= x - \langle x, \alpha^\vee \rangle \alpha + \langle x, \tilde{\alpha}^\vee \rangle \alpha = x - \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle \alpha. \end{aligned}$$

If the formula holds for some $n \geq 1$ then

$$\begin{aligned} (s \circ t)^{n+1}(x) &= s(t(x - n \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle \alpha)) = s(t(x)) - n \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle s(t(\alpha)) \\ &= x - \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle \alpha - n \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle s(t(\alpha)) \\ &= x - (n+1) \langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle \alpha. \end{aligned}$$

Because s and t are automorphisms of V (as a k-vector space) the same goes for $s \circ t$. Because $(s \circ t)(R) = s(t(R)) \subseteq s(R) \subseteq R$ and R is finite it follows that the restriction $(s \circ t)|_R$ is a permutation of R. Since R is finite $(s \circ t)|_R$ has finite order, i.e. there exists some $n \geq 1$ with $(s \circ t)^n|_R = \mathrm{id}_R$. Because R spans V it already follows that $(s \circ t)^n = \mathrm{id}_V$. Together with (1) it follows that $\langle x, \alpha^\vee - \tilde{\alpha}^\vee \rangle = 0$ for every $x \in V$, so $\alpha^\vee - \tilde{\alpha}^\vee = 0$ as seen in Remark 4.1.2. ☐

Appendices

A. Hopf algebras

In this chapter we give the definition of k-algebras, k-coalgebra, k-bialgebras and Hopf algebras. For this we fix an arbitrary field k. We start by reintroducing the notion of a k-algebra in terms of diagrams, where we require all k-algebras to be unitary and associative. Then we dualize this concept to obtain the notion of a k-coalgebra. Combining the previous two we arrive at k-bilalgebras, which will then lead to definition of a Hopf algebra by equipping a k-bialgebra with an antipodel map.

Most of the definitions and diagrams in this chapter are taken from [**Brown**].

A.1. algebras, coalgebra and bialgebras

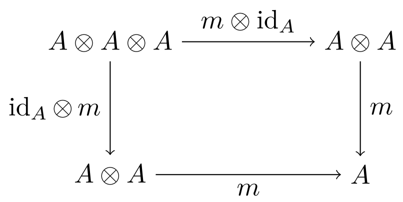
A.1.1. k-algebras

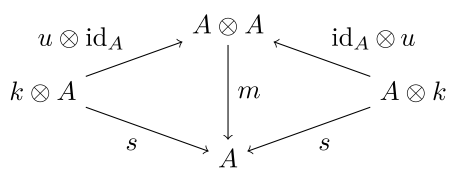
Definition A.1.1. For any k-vector spaces A and B let

$$\tau_{A,B} : A \otimes B \to B \otimes A, \quad a \otimes b \mapsto b \otimes a.$$

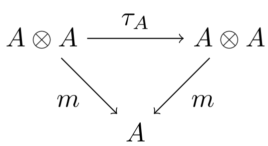
For any vector space A abbreviate $\tau_A \coloneqq \tau_{A,A}$.

Definition A.1.2. A k-algebra is a tupel (A, m, u) consisting of a k-vector space A, linear maps $m: A \otimes A \to A$, called the *multiplication* and a linear map $u: k \to A$, called the *unit* such that the following two diagrams commute, where s denotes the respective scalar multiplication.





The commutativity of the left diagram is the *associativity aximion* and the commutativity of the right diagram is the *unit axiom*. The k-algebra (A, m, u) is called *commutative* if the following diagram commutes:



Remark A.1.3. We will often just refer to a k-algebra (A, m, u) by A without explicitely mentioning m and u. If necessary we will then refer to the multiplication by m_A and the unit by u_A.

Remark A.1.4. Definition A.1.2 is equivalent to the usual definition of a k-algebra, namely a k-vector space A together with a bilinear map $\hat{m}\colon A \times A \to A, (a, b) \mapsto a \cdot b$ which is associative in the sense that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in A$, and such that a unit $1 \in A$ exists, i.e. an element for which $1 \cdot a = a \cdot 1 = a$ for every $a \in A$.

Given a k-algebra in the usual sense the multiplication $\hat{m}$ corresponds to a linear map $m\colon A \otimes A \to A, a \otimes b \mapsto a \cdot b$, and that m satisfies the associativity axiom is equivalent to $\hat{m}$ being associative. That a linear map $u\colon k \to A$ satisfies the unit axiom is equivalent to $u(1)$ being a unit in A. That $(A, \hat{m})$ is commutative is then also equivalent to (A, m, u) being commutative.

We will use both definitions, depending on which is more useful in a given situation, and switch between them if necessary.

Example A.1.5. Let G be any group. Then the *group algebra* $k[G]$ is defined by the underlying vector space being kG, the free vector space with basis G, and the multiplication which arises from extending the multiplication of G linearly, i.e.

$$\left(\sum_{g\in G} \lambda_g g\right) \cdot \left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} (\lambda_g \mu_h)(gh) = \sum_{g\in G} \left(\sum_{h\in G} \lambda_{gh^{-1}} \mu_h\right) g.$$

The associativity of the multiplication can be checked by direct calculation. The unit of the group algebra $k[G]$ is given by the identity of the group $e \in G \subseteq k[G]$. The group algebra $k[G]$ is commutative if and only if G is.

Example A.1.6. Let A and B be k-algebras. Then $A \otimes B$ carries the structure of a k-algebra via

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 \cdot a_2) \otimes (b_1 \cdot b_2) \quad \text{for all } a_1, a_2 \in A \text{ and } b_1, b_2 \in B.$$

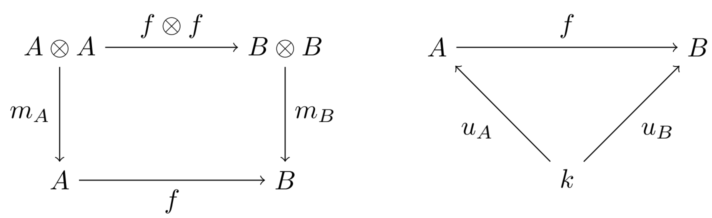
Then $1_{A\otimes B} = 1_A \otimes 1_B$. The multiplication can also be expressed by the equality

$$m_{A\otimes B}\colon A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes \tau_{A,B} \otimes \mathrm{id}_A} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

and the unit by

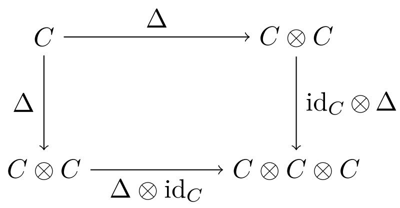
$$u_{A\otimes B}\colon k \xrightarrow{\lambda \mapsto \lambda \otimes 1} k \otimes k \xrightarrow{u_A \otimes u_B} A \otimes B$$

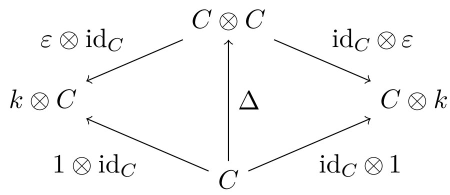
Definition A.1.7. Let A and B be k-algebras. A linear map $f\colon A \to B$ is a *homomorphism of k-algebras* if the following two diagrams commute:



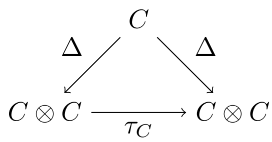
A.1.2. k-coalgebras

Definition A.1.8. A k-coalgebra is a tupel (C, Δ, ε) consisting of a k-vector space C, a linear map $\Delta\colon C \to C \otimes C$, called the *comultiplication* and a linear map $\varepsilon\colon C \to k$, called the *counit*, such that the following two diagrams commute:





The commutativity of the left diagram is the *coassociativity axiom* and the commutativity of the right diagram is the *counit axiom*. The coalgebra (C, Δ, ε) is called *cocommutative* if the following diagram commutes:



Examples A.1.9. 1. Let C be a k-vector space and $(x_i)_{i \in I}$ a basis of V. Then V carries the structure of a k-coalgebra via the comultiplication $\Delta\colon C \to C \otimes C$ defined by

$$\Delta(x_i) \coloneqq x_i \otimes x_i \quad \text{for every } i \in I$$

and the counit $\varepsilon\colon C \to k$ defined by

$$\varepsilon\left(\sum_{i \in I} \lambda_i x_i\right) \coloneqq \sum_{i \in I} \lambda_i.$$

To see that Δ is coassociative notice that for every $i \in I$

$$\begin{aligned} ((\mathrm{id}_C \otimes \Delta) \circ \Delta)(x_i) &= (\mathrm{id}_C \otimes \Delta)(x_i \otimes x_i) = x_i \otimes x_i \otimes x_i \\ &= (\Delta \otimes \mathrm{id}_C)(x_i \otimes x_i) = ((\Delta \otimes \mathrm{id}_C) \circ \Delta)(x_i). \end{aligned}$$

To notice that ε is a counit notice that for every $i \in I$

$$((\varepsilon \otimes \mathrm{id}_C) \circ \Delta)(x_i) = (\varepsilon \otimes \mathrm{id}_C)(x_i \otimes x_i) = 1 \otimes x_i = (1 \otimes \mathrm{id}_C)(x_i)$$

and similarly $(\mathrm{id}_C \otimes \varepsilon) \circ \Delta = \mathrm{id}_C \otimes 1$.

2. At a special case of the above example it follows that for any group G the group algebra $k[G]$ also carries the structure of a k-coalgebra via the comultiplication $\Delta\colon k[G] \to k[G] \otimes k[G], g \mapsto g \otimes g$ and counit $\varepsilon\colon k[G] \to k, \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g$.

Example A.1.10. If C and D are k-coalgebras then $C \otimes D$ is a also a k-coalgebra via the comultiplication

$$\Delta_{C\otimes D} \colon C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C \otimes \tau_{C,D} \otimes \mathrm{id}_C} C \otimes D \otimes C \otimes D$$

and the counit

$$\varepsilon_{C\otimes D} \colon C \otimes D \xrightarrow{\varepsilon_C \otimes \varepsilon_D} k \otimes k \xrightarrow{\lambda\otimes\mu \mapsto \lambda\mu} k.$$

Definition A.1.11. Let C and D be k-coalgebras. A linear map $f \colon C \to D$ is a homomorphism of k-coalgebras if the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow[f \otimes f]{} & D \otimes D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle \varepsilon_C}\searrow \quad \swarrow{\scriptstyle \varepsilon_D} & \\ & k & \end{array}$$

B. Schur's Lemma

Unless otherwise noted k always is some arbitrary field. Whenever we talk about a ring (resp. k-algebra) we always mean an associative and unitary one, and homomorphisms of rings (resp. k-algebras) are required to respect the unit. We assume that the reader is familiar with the definition of a module over a ring notion of a submodules. By an (left) R-module M over a ring R we always mean an unitial module, i.e. $1 \cdot m = m$ for every $m \in M$.

B.1. Classic version

Definition B.1.1. Let M be a module over a ring R. Then M is called *simple* or *irreducible* if M contains precisely two sumbodules. Equivalently M in nonzero and its only submodules are the *trivial* ones, namely 0 and M itself.

Lemma B.1.2 (Schur)**.** *Let R be a ring and M a simple module over R. Then any endomorphism of modules $f\colon M \to M$ is either zero or an isomorphism. In particular $\operatorname{End}_R(M)$ is a skew field.*

Proof. As M is nonzero f cannot be zero and an isomorphism at the same time. If $f \neq 0$ then $\ker f$ is a proper submodule of M and $\operatorname{im} f$ is a nonzero submodule of M, so $\ker f = 0$ and $\operatorname{im} f = M$ because M is simple. □

Corollary B.1.3. *Let M be an A-module over a k-algebra A. Then $\operatorname{End}_A(M)$ is a division algebra over k.*

Lemma B.1.4. *Let D be a division algebra over an algebraically closed field k. If $x \in D$ is algebraic over k then already $x \in k$.*

Proof. Let $P \in k[T]$ be nonzero with $P(x) = 0$. W.l.o.g. P can assumed to be monic. Because k is algebraically closed there exist $\alpha_1, \dots, \alpha_r \in k$ with $P = \prod_{i=1}^{r}(x - \alpha_i)$. Because $0 = P(x) = c\prod_{i=1}^{n}(x - \alpha_i)$ and D is a skew field it follows that $x = \alpha_i$ for some i and therefore $x \in k$. □

Corollary B.1.5. *Let k be an algebraically closed field and L a finite-dimensional division algebra over k. Then $L = k$.*

Proof. Let $x \in L$. Because L is finite-dimenisonal over k there exists some $n \geq 1$ such that $1, x, x^2, \dots, x^n$ are linearly dependent over k. Therefore there exist some $a_0, a_1, \dots, a_n \in k$ such that $a_0 + a_1 x + \cdots + a_n x^n = 0$ is a non-trivial linear combination. Then $P = \sum_{i=0}^{n} a_i T^i \in k[T]$ is nonzero with $P(x) = 0$, so x is algebraic over k. From Lemma B.1.4 it follows that $x \in k$. □

Corollary B.1.6 (Schur, classic Version)**.** *Let k be an algebraically closed field and M a simple A-module for a k-algebra A. If M is finite-dimensional over k then* $\operatorname{End}_A(M) = k$*, i.e. every module endomorphism of M is given by multiplication with a scalar.*

Corollary B.1.7. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k and V a irreducible and finite-dimensional representation of $\mathfrak{g}$. Then* $\operatorname{End}_{\mathfrak{g}}(V) = k$*, i.e. every endomorpism of V as a representation of $\mathfrak{g}$ is given by an multiplication with some scalar.*

Proof. Take V as a simple module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and apply Corollary B.1.6. □

B.2. Generalization by Dixmier

Definition B.2.1. Let V be a vector space over a field k. An endomorphism $\varphi \in \operatorname{End}_k(V)$ is called *algebraic* over k there exists some nonzero polynomial $P \in k[T]$ mit $P(\varphi) = 0$.

Lemma B.2.2. *Let k be an algebraically closed field, V a vector space over k and $D \subseteq \operatorname{End}_k(V)$ a division algebra over k. If $\varphi \in D$ is algebraic over k then $\varphi = \alpha \operatorname{id}_V$ for some $\alpha \in k$.*

Proof. This follows directly from Lemma B.1.4. □

Corollary B.2.3. *Let k be an algebraically closed field, A a k-algebra and M a simple A-module. If $\varphi \in \operatorname{End}_A(M)$ is algebraic then $\varphi = \alpha \operatorname{id}_M$ for some $\alpha \in k$.*

Proof. This follows directly from Lemma B.2.2 because $\operatorname{End}_A(M) \subseteq \operatorname{End}_k(M)$ is a division algebra over k by Lemma B.1.2. □

The following Proposition traces back to [**Dixmier**]. (At least this is what I found on the web — I could not find the original article, nor would I be able to read it (as it was apparently written in French)).

Proposition B.2.4 (Dixmier)**.** *Let M be a simple A-module for a k-algebra A, such that $\dim_k M > \operatorname{card} k$. Then every $\varphi \in \operatorname{End}_A(M)$ is algebraic over k.*

Proof. Suppose that there exists some $\varphi \in \operatorname{End}_A(M)$ which is not algebraic over k. Then the kernel of the map

$$\iota\colon k[T] \to \operatorname{End}_A(M), \quad P \mapsto P(\varphi)$$

is zero, hence ι is an inclusion of $k[T]$ into $\operatorname{End}_A(M)$, which is a skew field by Lemma B.1.2. It follows That ι can be extended to a well-defined inclusion

$$\theta\colon k(T) \to \operatorname{End}_A(M), \quad \frac{P}{Q} \mapsto P(\varphi)Q(\varphi)^{-1}.$$

Hence M carries the structure of a $k(T)$-vector space with

$$\frac{P}{Q} \cdot m = P(\varphi)Q(\varphi)^{-1}(m) \quad \text{for every } \frac{P}{Q} \in k(T) \text{ and } m \in M.$$

As M is a nonzero $k(T)$-vector space it follows that $\dim_k M \geq \dim_k k(T)$. To see this notice that if L/k is any field extension and V a nonzero L-vector space then there exists an inclusion $L \hookrightarrow V$ of L-vector spaces. This is then also an inclusion of k-vector spaces, which is why $\dim_k V \geq \dim_k L$. The statement follows with $L = k(T)$ and $V = M$. (This is a straightforward generalization of the fact that every complex nonzero vector space is at least twodimensional as a real vector space.) Since $(1/(T-a))_{a\in k}$ is a familiy of elements of $k(T)$ which is linearly independent over k it also follows that $\dim_k k(T) \geq \operatorname{card} k$.

Putting the above observations together it follows that

$$\dim_k M \geq \dim_k k(T) \geq \operatorname{card} k,$$

contradicting the assumption that $\operatorname{card} k > \dim_k M$. □

Corollary B.2.5. *Let k be an algebraically closed field, A a k-algebra and M a simple A-module. If $\operatorname{card} k > \dim_k M$ then $\operatorname{End}_A(M) = k$.*

Proof. This is a combination of Corollary B.2.3 and Proposition B.2.4. □

Corollary B.2.6. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k and V an irreducible representation of $\mathfrak{g}$ with $\operatorname{card} k > \dim_k V$. Then $\operatorname{End}_{\mathfrak{g}}(V) = k$.*

Proof. Take V as a simple module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and apply Corollary B.2.5. □

Example B.2.7. Let $\mathfrak{g}$ be complex Lie algebra and V an irreducible representation of $\mathfrak{g}$ of countable dimension. Then $\operatorname{End}_{\mathfrak{g}}(V) = \mathbb{C}$.

Remark B.2.8. The requirement that $\operatorname{card} k > \dim_k M$ in Corollary B.2.5 can not be dropped without adding some other restraints. To see this take $k := \overline{\mathbb{Q}}$ as well as $A = M = \overline{\mathbb{Q}}(T)$. Then $\dim_k M = \operatorname{card} k$ and $\operatorname{End}_A(M) = \operatorname{End}_{\overline{\mathbb{Q}}(T)}(\overline{\mathbb{Q}}(T)) = \overline{\mathbb{Q}}(T)$.

B.3. Generalizaton by Quillen

The following Proposition is due to [**Quillen**].

Proposition B.3.1 (Quillen)**.** *Let k be a field and A a filtered k-algebra, such that $\operatorname{gr} A$ is finitely generated and commutative as a k-algebra. If M is a simple A-module then every $\varphi \in \operatorname{End}_A(M)$ is algebraic over k.*

Corollary B.3.2. *Let $\mathfrak{g}$ be finite-dimensional Lie algebra over an algebraically closed field k and V as irreducible representation of $\mathfrak{g}$. Then $\operatorname{End}_{\mathfrak{g}}(V) = k$.*

Proof. Take V as a simple module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. If $x_1, \ldots, x_n$ is a k-basis of $\mathfrak{g}$ then by the abstract version of the PBW theorem

$$\operatorname{gr} \mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g}) \cong k[x_1, \ldots, x_n].$$

Applying Proposition B.3.1 to $\mathcal{U}(\mathfrak{g})$ and V the statement follows from Corollary B.2.3. □