

# **Representation Theory I**

**A short collection of fun facts about  
Lie algebras and their representations**

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# Preface

The following text is based on my notes for the lecture *Representation Theory I* which was given by Prof. Dr. Stroppel during the summer term 2015 at the University of Bonn. The lecture is mostly about the finite dimensional representation theory of finite dimensional semisimple complex Lie-Algebras. No previous knowledge about Lie-Algebras is required. At a few points I also took some motivation from [\[Hum72\]](#), mostly to clarify some proofs.

These notes have mainly two purposes: One is to prepare myself for the exam, as typing out these notes forces myself to go through them at a slow speed, paying much attention to details and look things up if necessary. The other is to allow the other students to learn with some nicely typed out text instead of a bunch of handwritten (and sometimes non-existing) notes.

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# 1. The Basics

## 1.1. Basic definitions

### 1.1.1. Definition and examples of Lie algebras

**Definition 1.1.1.** Let  $\mathfrak{g}$  be a vector space over some field  $k$ . A  $k$ -bilinear map

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

is called a *Lie bracket* if it satisfies the following two conditions:

1.  $[\cdot, \cdot]$  is *alternating*, i.e.  $[x, x] = 0$  for every  $x \in \mathfrak{g}$ .
2.  $[\cdot, \cdot]$  satisfies the *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

A  $k$ -vector space  $\mathfrak{g}$  together with the Lie-bracket  $[\cdot, \cdot]$  is called a  *$k$ -Lie algebra*.

**Remark 1.1.2.** Any Lie bracket  $[\cdot, \cdot]$  is antisymmetric, i.e.  $[y, x] = -[x, y]$  for all  $x, y \in \mathfrak{g}$ , because

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

**Remark 1.1.3.** Using the antisymmetry of the Lie bracket the Jacobi identity can be rewritten as

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad \text{for all } x, y, z \in \mathfrak{g}.$$

**Examples 1.1.4.** 1. Any vector space  $V$  becomes a Lie algebra via

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

2. Any *associative*  $k$ -algebra  $A$  becomes a Lie algebra via

$$[a, b] = ab - ba \quad \text{for all } a, b \in A.$$

It is clear that  $[\cdot, \cdot]$  is alternating and because  $A$  is associative it follows that for all  $a, b, c \in A$

$$\begin{aligned} & [a, [b, c]] + [b, [c, a]] + [c, [a, b]] \\ &= [a, (bc - cb)] + [b, (ca - ac)] + [c, (ab - ba)] \\ &= a(bc - cb) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c \\ &= abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\ &= 0. \end{aligned}$$

The following two are important examples of this construction:

a) The  $k$ -algebra of  $(n \times n)$ -matrices  $M_n(k)$  becomes a Lie algebra via

$$[A, B] = AB - BA \quad \text{for all } A, B \in M_n(k).$$

This is called the *general linear Lie algebra* and is denoted by  $\mathfrak{gl}_n(k)$ .

b) More generally for any  $k$ -vector space the  $k$ -algebra  $\text{End}_k(V)$  becomes a Lie algebra via

$$[\varphi_1, \varphi_2] := \varphi_1 \circ \varphi_2 - \varphi_2 \circ \varphi_1 \quad \text{for all } \varphi_1, \varphi_2 \in \text{End}_k.$$

This is called the *general linear Lie algebra for  $V$*  and is denoted by  $\mathfrak{gl}(V)$ .

**Definition 1.1.5.** Let  $\mathfrak{g}$  be a  $k$ -Lie algebra. A *Lie subalgebra* of a  $\mathfrak{g}$  is a  $k$ -linear subspace  $\mathfrak{h} \subseteq \mathfrak{g}$  such that

$$[x, y] \in \mathfrak{h} \quad \text{for all } x, y \in \mathfrak{h}.$$

An *ideal* inside  $\mathfrak{g}$  is a  $k$ -linear subspace  $I \subseteq \mathfrak{g}$  such that

$$[x, y] \in I \quad \text{for all } x \in \mathfrak{g} \text{ and } y \in I.$$

That  $I$  is an ideal in  $\mathfrak{g}$  is denoted by  $I \trianglelefteq \mathfrak{g}$ .

**Remark 1.1.6.** It is not necessary to distinguish between left ideals or right ideals in a Lie algebra because the Lie bracket is antisymmetric.

**Remark 1.1.7.** For a Lie algebra  $\mathfrak{g}$  and a subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  it follows that  $\mathfrak{h}$  becomes a Lie algebra by restricting the Lie bracket of  $\mathfrak{g}$  to  $\mathfrak{h}$ . Every ideal inside  $\mathfrak{g}$  is also a subalgebra of  $\mathfrak{g}$ .

**Definition 1.1.8.** Let  $\mathfrak{g}$  be a Lie algebra. The *center* of  $\mathfrak{g}$  is

$$Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for every } y \in \mathfrak{g}\}$$

$\mathfrak{g}$  is called *abelian* if  $Z(\mathfrak{g}) = 0$ , i.e. if  $[x, y] = 0$  for all  $x, y \in \mathfrak{g}$ .

**Definition 1.1.9.** For a Lie-algebra  $\mathfrak{g}$  over some field  $k$  and subsets  $X, Y \subseteq \mathfrak{g}$  let

$$[X, Y] := \text{span}_k\{[x, y] \mid x \in X, y \in Y\}.$$

**Remark 1.1.10.** Clearly  $\mathfrak{g}$  is abelian if and only if  $[\mathfrak{g}, \mathfrak{g}] = 0$ . Also notice that  $[\mathfrak{g}, \mathfrak{g}]$  and  $Z(\mathfrak{g})$  are ideals inside  $\mathfrak{g}$ .

**Lemma 1.1.11.** Let  $\mathfrak{g}$  be a Lie algebra over some field  $k$ .

1. If  $I_\lambda, \lambda \in \Lambda$  is a collection of ideals  $I_\lambda \trianglelefteq \mathfrak{g}$  then also  $\bigcap_{\lambda \in \Lambda} I_\lambda \trianglelefteq \mathfrak{g}$  and  $\sum_{\lambda \in \Lambda} I_\lambda \trianglelefteq \mathfrak{g}$ .
2. If  $I, J \trianglelefteq \mathfrak{g}$  then also  $[I, J] \trianglelefteq \mathfrak{g}$ .

*Proof.* 1. This follows from direct calculation.

2. As  $[I, J]$  is spanned by the elements  $[y, z]$  with  $y \in I$  and  $z \in J$  it is enough to show that  $[x, [y, z]] \in [I, J]$  for every  $x \in \mathfrak{g}$ ,  $y \in I$  and  $z \in J$ . This follows from  $I, J \trianglelefteq \mathfrak{g}$  and the Jacobi identity, because

$$[x, [y, z]] = \underbrace{[[x, y], z]}_{\in I} + \underbrace{[y, [x, z]]}_{\in J} \in [I, J]. \quad \square$$

**Definition 1.1.12.** A Lie algebra  $\mathfrak{g}$  is called *linear* if  $\mathfrak{g}$  is a Lie subalgebra of  $\mathfrak{gl}(V)$  for some finite dimensional vector space  $V$ .

**Example 1.1.13.** 1. Let  $\mathfrak{g} = \mathfrak{gl}_n(k)$ . Then

$$\mathfrak{sl}_n(k) = \{A \in \mathfrak{g} \mid \operatorname{tr} A = 0\}$$

is an ideal in  $\mathfrak{g}$  because

$$\mathfrak{sl}_n(k) = [\mathfrak{g}, \mathfrak{g}].$$

To see this first notice that on the one hand

$$\operatorname{tr}[A, B] = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = \operatorname{tr}(AB) - \operatorname{tr}(AB) = 0$$

for all  $A, B \in \mathfrak{g}$  and therefore  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{sl}_n(k)$ .

On the other hand notice that  $\mathfrak{sl}_n(k)$  has a basis given by the elementary matrices  $e_{ij}$  with  $1 \leq i \neq j \leq n$  and  $e_{11} - e_{ii}$  with  $i = 2, \dots, n$ . Each of these matrices is given as a commutator, namely  $e_{ij} = [e_{ii}, e_{ij}]$  for  $1 \leq i \neq j \leq n$  and  $e_{11} - e_{ii} = [e_{1i}, e_{i1}]$  for  $i = 2, \dots, n$ . Therefore  $\mathfrak{sl}_n(k) \subseteq [\mathfrak{g}, \mathfrak{g}]$ .

2. The upper triangular matrices

$$\mathfrak{t}_n(k) := \left\{ \left( \begin{array}{cccc} a_{11} & \cdots & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{array} \right) \mid a_{ij} \in k \text{ for every } 1 \leq i \leq j \leq n \right\}$$

are a Lie subalgebra of  $\mathfrak{gl}_n(k)$ .

3. The strictly upper triangular matrices

$$\mathfrak{n}_n(k) := \left\{ \left( \begin{array}{cccc} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{array} \right) \mid a_{ij} \in k \text{ for every } 1 \leq i < j \leq n \right\}$$

are a Lie subalgebra of  $\mathfrak{t}_n(k)$  and therefore also of  $\mathfrak{gl}_n(k)$ . It is even an ideal in  $\mathfrak{t}_n(k)$  because  $\mathfrak{n}_n(k) = [\mathfrak{t}_n(k), \mathfrak{t}_n(k)]$ .

**Definition 1.1.14.** If  $\mathfrak{g}$  is a Lie algebra and  $U \subseteq \mathfrak{g}$  a linear subspace then

$$N_{\mathfrak{g}}(U) := \{x \in \mathfrak{g} \mid [x, y] \in \mathfrak{g} \text{ for every } y \in U\}$$

is called the *normalizer* of  $U$  in  $\mathfrak{g}$  and

$$Z_{\mathfrak{g}}(U) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for every } y \in U\}$$

is called the *centralizer* of  $U$  in  $\mathfrak{g}$ . For a single element  $x \in \mathfrak{g}$  the centralizer of  $x$  in  $\mathfrak{g}$  defined as

$$Z_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [x, y] = 0\}.$$

**Lemma 1.1.15.** Let  $\mathfrak{g}$  be a Lie algebra and  $U \subseteq \mathfrak{g}$  a linear subspace. Then  $N_{\mathfrak{g}}(U)$  and  $Z_{\mathfrak{g}}(U)$  are Lie subalgebras of  $\mathfrak{g}$ .  $Z_{\mathfrak{g}}(x)$  is a Lie subalgebra of  $\mathfrak{g}$  for every  $x \in \mathfrak{g}$ .

*Proof.* If  $x, y \in N_{\mathfrak{g}}(U)$  then by the Jacobi identity it follows for every  $z \in U$  that

$$[[x, y], z] = -[z, [x, y]] = -[[z, x], y] - [x, [z, y]] \in \mathfrak{h}$$

and therefore  $[x, y] \in N_{\mathfrak{g}}(U)$ . In the same way it follows for all  $x, y \in \mathfrak{g}$  and  $z \in U$  that

$$[[x, y], z] = -[[z, x], y] - [x, [z, y]] = 0$$

and therefore  $[x, y] \in Z_{\mathfrak{g}}(U)$ . For every  $x \in \mathfrak{g}$  the span  $kx \subseteq \mathfrak{g}$  is a linear subspace with  $Z_{\mathfrak{g}}(x) = Z_{\mathfrak{g}}(kx)$ , which is why  $Z_{\mathfrak{g}}(x)$  is a Lie subalgebra of  $\mathfrak{g}$ .  $\square$

**Remark 1.1.16.** Let  $\mathfrak{g}$  be a Lie algebra and  $L \subseteq \mathfrak{g}$  a linear subspace. Then  $L$  is a Lie subalgebra if and only if  $L \subseteq N_{\mathfrak{g}}(L)$ . Then  $L$  is not only contained in  $N_{\mathfrak{g}}(L)$  but  $N_{\mathfrak{g}}(L)$  is the maximal subalgebra of  $\mathfrak{g}$  which contains  $L$  as an ideal. In particular  $L$  is an ideal in  $\mathfrak{g}$  if and only if  $N_{\mathfrak{g}}(L) = \mathfrak{g}$ .

### 1.1.2. Homomorphisms of Lie algebras

**Definition 1.1.17.** Given Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  over the same field  $k$  a *homomorphism of Lie algebras*  $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a  $k$ -linear map  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  such that

$$f([x, y]) = [f(x), f(y)] \quad \text{for all } x, y \in \mathfrak{g}_1.$$

**Examples 1.1.18.** 1. For any Lie algebra  $\mathfrak{g}$  the identity  $\text{id}_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathfrak{g}$  is a Lie algebra homomorphism.

2. Given Lie algebras  $\mathfrak{g}_1, \mathfrak{g}_2$  and  $\mathfrak{g}_3$  and Lie algebra homomorphisms  $f_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  and  $f_2: \mathfrak{g}_2 \rightarrow \mathfrak{g}_3$  the composition  $f_2 \circ f_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_3$  is also a homomorphism of Lie algebras.

3. If  $\mathfrak{g}$  is a Lie algebra and  $\mathfrak{h} \subseteq \mathfrak{g}$  a Lie subalgebra then the inclusion  $\mathfrak{h} \hookrightarrow \mathfrak{g}$  is a homomorphism of Lie algebras.

4. Given two abelian Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  any linear map  $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is already a homomorphism of Lie algebras.



5. Let  $\mathfrak{g}$  be a Lie algebra over an arbitrary field  $k$ . Then for every  $x \in \mathfrak{g}$  let

$$\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g} \quad \text{mit} \quad \text{ad}(x)(y) = [x, y] \quad \text{for every } y \in \mathfrak{g}.$$

Then the map  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is a homomorphism of Lie algebras. This follows from the Jacobi identity because for all  $x, y, z \in \mathfrak{g}$

$$\begin{aligned} \text{ad}([x, y])(z) &= [[x, y], z] = -[z, [x, y]] = -[[z, x], y] - [x, [z, y]] = [x, [y, z]] - [y, [x, z]] \\ &= \text{ad}(x)\text{ad}(y)(z) - \text{ad}(y)\text{ad}(x)(z) = [\text{ad}(x), \text{ad}(y)](z). \end{aligned}$$

6. If  $A_1$  and  $A_2$  are associative  $k$ -algebras and  $f: A_1 \rightarrow A_2$  a homomorphism of  $k$ -algebras then it is also a homomorphism of Lie algebras because

$$f([a, b]) = f(ab - ba) = f(a)f(b) - f(b)f(a) = [f(a), f(b)] \quad \text{for all } a, b \in A_1.$$

7. Let  $\mathfrak{g}$  be a Lie algebra over an arbitrary field  $k$ . If  $\phi: \mathfrak{sl}_2(k) \rightarrow \mathfrak{g}$  is a homomorphism of Lie algebras then the images

$$E := \phi(e), \quad H := \phi(h), \quad F := \phi(f)$$

satisfy the relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

On the other hand every triple  $(E', H', F')$  of elements satisfying the relations above (with  $X$  replaced by  $X'$  for  $X \in \{E, H, F\}$ ) gives rise to a unique homomorphism of Lie algebras  $\phi': \mathfrak{sl}_2(k) \rightarrow \mathfrak{g}$  with

$$\phi'(E) = E', \quad \phi'(H) = H', \quad \phi'(F) = F'.$$

Hence there is a bijection between Lie algebra homomorphisms  $\mathfrak{sl}_2(k) \rightarrow \mathfrak{g}$  and triples as above. Such triples will play an important part later on.

**Definition 1.1.19.** Let  $\mathfrak{g}_1, \mathfrak{g}_2$  be Lie algebras over the same field  $k$ . A homomorphism of Lie algebras  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is called an *isomorphism of  $k$ -Lie algebras* if  $f$  is bijective.

**Lemma 1.1.20.** *If  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is an isomorphism of  $k$ -Lie algebras, then the  $k$ -linear map  $f^{-1}: \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$  is also a homomorphism of Lie-algebras and therefore also an isomorphism.*

*Proof.* For all  $x, y \in \mathfrak{g}_2$

$$\begin{aligned} f^{-1}([x, y]) &= f^{-1}([f(f^{-1}(x)), f(f^{-1}(y))]) \\ &= f^{-1}(f([f^{-1}(x), f^{-1}(y)])) \\ &= [f^{-1}(x), f^{-1}(y)]. \end{aligned} \quad \square$$

**Remark 1.1.21.** It follows that Lie algebras over the same field  $k$  together with homomorphisms of Lie algebras and their usual composition form a category, which will be denoted by  $k\text{-Lie}$ . An *isomorphism of  $k$ -Lie algebras* is the same as an isomorphism in  $k\text{-Lie}$ .

**Example 1.1.22** (Classification of one- and two-dimensional Lie algebras). Let  $k$  be any field.

As every linear map between abelian Lie algebras is already a homomorphism of Lie algebras it follows that there exists up to isomorphism exactly one  $n$ -dimensional abelian Lie algebra over  $k$  for every  $n \in \mathbb{N}$ .

If  $\mathfrak{g}$  is a one-dimensional Lie algebra over  $k$  then the Lie bracket of  $\mathfrak{g}$  is zero because it is alternating, which is why  $\mathfrak{g}$  is abelian. Hence there is up to isomorphism exactly one one-dimensional Lie algebra over  $k$ .

Up to isomorphism there exists exactly one two-dimensional abelian  $k$ -Lie algebra.

Suppose that  $\mathfrak{g}$  is a two-dimensional, non-abelian Lie algebra over  $k$ . Let  $\tilde{x}, \tilde{y}$  be a basis of  $\mathfrak{g}$ . Because  $[\mathfrak{g}, \mathfrak{g}]$  is non-abelian it follows that  $[\mathfrak{g}, \mathfrak{g}] \neq 0$  and on the other hand  $[\mathfrak{g}, \mathfrak{g}]$  is spanned by  $[\tilde{x}, \tilde{y}]$  because the Lie bracket is alternating, so  $[\mathfrak{g}, \mathfrak{g}] = k[\tilde{x}, \tilde{y}]$  with  $[\tilde{x}, \tilde{y}] \neq 0$ . Let  $x := [\tilde{x}, \tilde{y}]$  and  $y \in \mathfrak{g}$  such that  $x, y$  is a basis of  $\mathfrak{g}$ . Then  $[x, y] \neq 0$  and  $[x, y] \in kx$ . By rescaling  $y$  it can be assumed that  $[x, y] = x$ . This that up to isomorphism there is at most one two-dimensional, non-abelian Lie algebra  $\mathfrak{g}$  over  $k$ .

Such an Lie algebra also exists. It can be realized as a subalgebra of  $\mathfrak{gl}_2(k)$  by choosing

$$x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_{12} \quad \text{and} \quad y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = e_{22}$$

because

$$[x, y] = [e_{12}, e_{22}] = e_{12}e_{22} - e_{22}e_{12} = e_{12} = x.$$

Hence there are up to isomorphism exactly two two-dimensional Lie algebras over  $k$ .

**Proposition 1.1.23** (Homomorphism theorem). *Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be Lie algebras and  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  a homomorphism of Lie algebras.*

1.  $\ker f \trianglelefteq \mathfrak{g}_1$  is an ideal.
2.  $\text{im } f \subseteq \mathfrak{g}_2$  is a Lie subalgebra.
3. If  $I \trianglelefteq \mathfrak{g}_1$  is an ideal with  $\ker f \subseteq I$  then there exists a unique homomorphism of Lie algebras  $\tilde{f}: \mathfrak{g}_1/I \rightarrow \mathfrak{g}_2$  with  $f = \tilde{f} \circ \pi$  where  $\pi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/I$  is the canonical projection.

$$\begin{array}{ccc} \mathfrak{g}_1 & & \\ \downarrow \pi & \searrow f & \\ \mathfrak{g}_1/I & \xrightarrow{\exists! \tilde{f}} & \mathfrak{g}_2 \end{array}$$

4. If  $I, J \trianglelefteq \mathfrak{g}$  are subideals with  $I \subseteq J$  then  $J/I \trianglelefteq \mathfrak{g}/I$  and the natural isomorphism of vector spaces

$$(\mathfrak{g}/I)/(J/I) \rightarrow \mathfrak{g}/I, \quad (x+I) + (J/I) \mapsto x+I$$

is already a natural isomorphism of Lie algebras.

5. If  $I, J \trianglelefteq \mathfrak{g}$  are subideals then the natural isomorphism of vector spaces

$$(I+J)/J \rightarrow I/(I \cap J)$$

defined by

$$(x+J) + I \mapsto x + (I \cap J) \quad \text{for every } x \in I$$

is already a natural isomorphism of Lie algebras.

**Remark 1.1.24.** For a Lie algebra  $\mathfrak{g}$  the ideal  $[\mathfrak{g}, \mathfrak{g}]$  is the minimal ideal inside  $\mathfrak{g}$  such that  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  is abelian. Furthermore given any abelian Lie algebra  $\mathfrak{h}$  any homomorphism of Lie algebras  $\mathfrak{g} \rightarrow \mathfrak{h}$  factorizes through a unique homomorphism of Lie algebras  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \rightarrow \mathfrak{h}$ .

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad} & \mathfrak{h} \\ & \searrow & \nearrow \exists! \\ & \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] & \end{array}$$

### 1.1.3. New Lie algebras from old ones

**Definition 1.1.25.** Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be Lie algebras over the same field  $k$ . Then the *product* of  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  is defined as the  $k$ -vector space  $\mathfrak{g}_1 \times \mathfrak{g}_2$  together with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]) \quad \text{for all } (x_1, y_1), (x_2, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

**Definition 1.1.26.** Let  $\mathfrak{g}$  be a Lie algebra and  $I \trianglelefteq \mathfrak{g}$ . Then the induced Lie algebra structure on the quotient vector space  $\mathfrak{g}/I$  is given by

$$[x+I, y+I] = [x, y] + I \quad \text{for all } x, y \in \mathfrak{g}.$$

**Remark 1.1.27.** The Lie algebra structure on the quotient  $\mathfrak{g}/I$  is well-defined: If  $x, y, x', y' \in \mathfrak{g}$  with  $x+I = x'+I$  and  $y+I = y'+I$  then  $x-x' \in I$  and  $y-y' \in I$  and thus

$$\begin{aligned} [x, y] + I &= [x' + x - x', y' + y - y'] + I \\ &= [x', y'] + \underbrace{[x', y - y']}_{\in I} + \underbrace{[x - x', y']}_{\in I} + \underbrace{[x - x', y - y']}_{\in I} + I = [x', y'] + I. \end{aligned}$$

The additional properties of a Lie bracket follows from the Lie bracket of  $\mathfrak{g}$  satisfying them.

**Lemma 1.1.28.** 1. If  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are Lie algebras then the canonical projections

$$\pi_i: \mathfrak{g}_1 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_i, \quad (x_1, x_2) \mapsto x_i \quad \text{for } i = 1, 2$$

are homomorphisms of Lie-algebras.

2. If  $\mathfrak{g}$  is a Lie algebra and  $I \trianglelefteq \mathfrak{g}$  then the canonical projection  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I, x \mapsto [x]$  is a homomorphism of Lie algebras.

**Lemma 1.1.29.** Let  $\mathfrak{g}$  be a Lie algebra over  $k$  and  $A$  an associative, commutative  $k$ -algebra. Then  $A \otimes_k \mathfrak{g}$  is a Lie algebra over  $k$  via

$$[a \otimes x, b \otimes y] = (ab) \otimes [x, y] \quad \text{for all } a, b \in A \text{ and } x, y \in \mathfrak{g}.$$

Similarly  $\mathfrak{g} \otimes_k A$  carries the structure of a Lie algebra over  $k$  via

$$[x \otimes a, y \otimes b] = [x, y] \otimes (ab) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

**Example 1.1.30.** If  $L/k$  is a field extension and  $\mathfrak{g}$  a Lie algebra over  $k$ , then  $L \otimes_k \mathfrak{g}$  is a Lie algebra over  $k$  via

$$[\lambda \otimes x, \mu \otimes y] = (\lambda\mu) \otimes [x, y] \quad \text{for alle } \lambda, \mu \in k \text{ and } x, y \in \mathfrak{g}.$$

$L \otimes_k \mathfrak{g}$  also carries the structure of an  $L$ -vector space via extension of scalars, i.e.

$$\lambda \cdot (\mu \otimes x) = (\lambda\mu) \otimes x \quad \text{for alle } \lambda, \mu \in k \text{ and } x \in \mathfrak{g},$$

and the Lie bracket is not only  $k$ -bilinear, but also  $L$ -bilinear. Hence the structure of a  $k$ -Lie algebra on  $L \otimes_k \mathfrak{g}$  can be extended to the structure of a Lie algebra over  $L$ . (Notice that the Jacobi-Identity is independent of the ground field.)

**Definition 1.1.31.** Let  $\mathfrak{g}$  be a Lie algebra and  $A = k[t, t^{-1}]$  be the algebra of Laurent polynomials over  $k$ . Then

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_k A$$

with the Lie bracket as in Lemma 1.1.29 is called the *loop (Lie) algebra* of  $\mathfrak{g}$ .

Another example for constructing new Lie algebras out of old ones are *central extensions*: Let  $\mathfrak{g}$  be any  $k$ -Lie algebra.

$$\tilde{\mathfrak{g}} := \mathfrak{g} \oplus k = \{x + \lambda c \mid x \in \mathfrak{g}, \lambda \in k\},$$

where we understand  $c$  as a formal variable. Suppose that  $\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  is a  $k$ -bilinear map satisfying the following properties:

1.  $\kappa$  is antisymmetric, i.e.  $\kappa(x, y) = -\kappa(y, x)$  for all  $x, y \in \mathfrak{g}$ .
2.  $\kappa$  satisfies the 2-cocycle condition

$$\kappa([x, y], z) + \kappa([y, z], x) + \kappa([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then  $\tilde{\mathfrak{g}}$  becomes a Lie algebra via

$$[x + \lambda c, y + \mu c] := [x, y] + \kappa(x, y)\lambda\mu c \quad \text{for all } x, y \in \mathfrak{g} \text{ and } \lambda, \mu \in k.$$

Note that  $c$  is central in  $\tilde{\mathfrak{g}}$  in the sense that  $[x, c] = 0$  for all  $x \in \mathfrak{g}$ .

**Example 1.1.32.** Let  $\mathfrak{g} = \mathfrak{gl}_n(k)$ . We define a symmetric bilinear form on  $\mathfrak{g}$  via

$$(A, B)_{\text{tr}} = \text{tr}(AB) \quad \text{for all } A, B \in \mathfrak{g}.$$

We define a bilinear form

$$\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow k[t, t^{-1}], \quad (x \otimes p, y \otimes q) \mapsto (x, y)_{\text{tr}} pq$$

We now get a 2-cocycle  $\kappa: \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow k$  via

$$\kappa(a, b) := \text{Res} \left( \frac{\partial a}{\partial t}, b \right).$$

$\kappa$  is also antisymmetric: Let  $a = x \otimes t^i$  and  $b = y \otimes t^j$  with  $x, y \in \mathfrak{g}$  and  $i, j \in \mathbb{Z}$ . Then

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \text{Res}(ix \otimes t^{i-1}, y \otimes t^j) = \text{Res}(it^{i+j-1}(x, y)_{\text{tr}}) \\ &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the same way we find that

$$\kappa(y \otimes t^j, x \otimes t^i) = \begin{cases} j(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $(\cdot, \cdot)_{\text{tr}}$  is symmetric we find that

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -j(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= -\kappa(y \otimes t^j, x \otimes t^i). \end{aligned}$$

#### 1.1.4. Derivations

**Definition 1.1.33.** Let  $A$  be a  $k$ -algebra (not necessarily unitary or even associative).

A *derivation of  $A$*  is a  $k$ -linear map  $d: A \rightarrow A$  such that

$$d(ab) = d(a)b + ad(b) \quad \text{for all } a, b \in A.$$

We set

$$\text{Der}(A) := \{d: A \rightarrow A \mid d \text{ is a derivation of } A\}.$$

**Remark 1.1.34.**  $\text{Der}(A)$  is a  $k$ -linear subspace of  $\text{End}_k(A)$ .

**Example 1.1.35.** Let  $A$  be a  $k$ -algebra. It follows from direct calculation that for all  $d, d' \in \text{Der}(A)$  the commutator  $[d, d'] = d \circ d' - d' \circ d$  is again a derivation  $\text{Der}(A)$ . Hence  $\text{Der}(A)$  is a Lie subalgebra of  $\mathfrak{gl}(A)$ .

**Lemma 1.1.36.** Let  $\mathfrak{g}$  be a Lie algebra. Then for any  $x \in \mathfrak{g}$  the map

$$\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g}, \quad y \mapsto [x, y]$$

is a derivation of  $\mathfrak{g}$ .

*Proof.* By the Jacobi identity

$$\begin{aligned} \text{ad}(x)([y, z]) &= [x, [y, z]] = [[x, y], z] + [y, [x, z]] \\ &= [\text{ad}(x)(y), z] + [y, \text{ad}(x)(z)] \end{aligned}$$

for all  $y, z \in \mathfrak{g}$ . □

**Definition 1.1.37.** Let  $\mathfrak{g}$  be a Lie algebra. A derivation of  $\mathfrak{g}$  is called *inner* if it is of the form  $\text{ad}(x)$  for some  $x \in \mathfrak{g}$ .

**Lemma 1.1.38.** If  $\mathfrak{g}$  is a Lie algebra then the inner derivations form an ideal inside of  $\text{Der}(\mathfrak{g})$ .

*Proof.* Let  $I := \text{im ad} \subseteq \text{Der}(\mathfrak{g})$  be the linear subspace of inner derivations. For any  $\delta \in \text{Der}$  and  $x \in \mathfrak{g}$  it follows that for any  $y \in \mathfrak{g}$

$$\begin{aligned} [\delta, \text{ad}(x)](y) &= (\delta \text{ad}(x) - \text{ad}(x)\delta)(y) \\ &= \delta([x, y]) - [x, \delta(y)] = [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)] \\ &= [\delta(x), y] = \text{ad}(\delta(x))(y). \end{aligned}$$

Hence  $[\delta, \text{ad}(x)] = \text{ad}(\delta(x)) \in I$ . □

### 1.1.5. Simple Lie algebras

**Definition 1.1.39.** A Lie algebra  $\mathfrak{g}$  is *simple* if  $0$  and  $\mathfrak{g}$  are the only ideals inside  $\mathfrak{g}$  and  $\mathfrak{g}$  is not abelian.

**Lemma 1.1.40.** Let  $\mathfrak{g}$  be a simple Lie algebra. Then  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  and  $Z(\mathfrak{g}) = 0$ .

*Proof.* Because  $\mathfrak{g}$  is simple it is not abelian. Therefore  $[\mathfrak{g}, \mathfrak{g}] \neq 0$  and  $Z(\mathfrak{g}) \neq \mathfrak{g}$ . Since  $[\mathfrak{g}, \mathfrak{g}]$  and  $Z(\mathfrak{g})$  are ideals inside  $\mathfrak{g}$  it follows that  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  and  $Z(\mathfrak{g}) = 0$ . By the homomorphism theorem  $\mathfrak{g}$  is isomorphic to its image  $\text{ad } \mathfrak{g}$  and hence to a linear Lie algebra. □

**Corollary 1.1.41.** Let  $\mathfrak{g}$  be simple. Then the homomorphism  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), x \mapsto \text{ad}(x)$  is injective. In particular  $\mathfrak{g}$  can be realized as a linear Lie algebra.

*Proof.* As part of Examples 1.1.18 has already been shown that  $\text{ad}$  is a homomorphism of Lie algebras. That it is injective follows directly from  $\ker \text{ad} = Z(\mathfrak{g}) = 0$ .  $\square$

It can be shown that every finite dimensional Lie algebra can be realized as a linear Lie algebra. This will not be proven in this lecture and is by far not trivial.

**Theorem 1.1.42 (Ado).** *Every finite dimensional Lie algebra  $\mathfrak{g}$  is isomorphic to a linear Lie algebra.*

**Examples 1.1.43.** 1. Since  $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] = \mathfrak{sl}_n(k) \neq \mathfrak{gl}_n(k)$  we find that  $\mathfrak{gl}_n(k)$  is not simple.

2. Let  $\mathfrak{g} = \mathfrak{sl}_2(k)$ . Then  $\mathfrak{g}$  is simple if and only if  $\text{char } k \neq 2$ . To see this consider the basis  $(e, h, f)$  of  $\mathfrak{sl}_2(k)$  consisting of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

of  $\mathfrak{sl}_2(k)$ . Then

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

If  $\text{char } k = 2$  then  $h$  spans a 1-dimensional ideal, thus  $\mathfrak{sl}_2(k)$  is not simple. Suppose that  $\text{char } k \neq 2$  and let  $I \subseteq \mathfrak{sl}_2(k)$  be an ideal with  $I \neq 0$ . It is clear that if  $I$  contains one of the basis vectors  $e, h$  or  $f$  it follows that  $I = \mathfrak{sl}_2(k)$ . Let  $x \in I$  with  $x \neq 0$  and write  $x = \alpha e + \beta h + \gamma f$ . Then

$$[e, x] = -2\beta e + \gamma h \quad \text{and} \quad [e, [e, x]] = -2\gamma e.$$

Since  $\gamma = 0$  or  $\gamma \neq 0$  we find that  $e \in I$ .

**Definition 1.1.44.** Let  $k$  be any field. The basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

of  $\mathfrak{sl}_2(k)$  is called the *standard basis* of  $\mathfrak{sl}_2(k)$ .

**Remark 1.1.45.** If  $\text{char } k = 0$  then  $\mathfrak{sl}_n(k)$  is simple for all  $n \geq 2$ .

## 1.2. Representations of Lie algebras

### 1.2.1. Definition and examples

**Definition 1.2.1.** Let  $\mathfrak{g}$  be a  $k$ -Lie algebra. A *representation* of  $\mathfrak{g}$  is a  $k$ -vector space  $V$  together with a homomorphism of Lie algebras  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ . This representation is called *faithful* if  $\rho$  is injective. The *dimension* of this representation is the dimension of  $V$ .

**Remark 1.2.2.** Equivalently a representation of  $\mathfrak{g}$  is a  $k$ -vector space  $V$  together with a  $k$ -bilinear map  $\mathfrak{g} \times V \rightarrow V, (x, v) \mapsto x.v$  such that

$$x.(y.v) - y.(x.v) = [x, y].v \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V. \quad (1)$$

Such an action results in a homomorphism of Lie algebras  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  by setting

$$\rho(x): V \rightarrow V, \quad v \mapsto x.v \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

On the other hand any homomorphism  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  an action as above by setting

$$x.v := \rho(x)(v) \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

Both constructions are inverse to each other.

We will not distinguish between these two concepts and choose whichever is more useful in the given situation.

**Remark 1.2.3.** If  $(x_i)_{i \in I}$  is a basis of a Lie algebra  $\mathfrak{g}$  then for  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  to be a homomorphism of Lie algebras it is enough to check that  $\rho([x_i, x_j]) = [\rho(x_i), \rho(x_j)]$  for all  $i, j \in I$ . Therefore it also suffices to check (1) for basis elements, i.e. that

$$x_i.(x_j.v) - x_j.(x_i.v) = [x_i, x_j].v \quad \text{for all } i, j \in I \text{ and } v \in V.$$

**Remark 1.2.4.** Ado's theorem is equivalent to every finite dimensional Lie algebra having a faithful representation.

**Examples 1.2.5.** 1. If  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  is a Lie subalgebra then  $V$  is a representation of  $\mathfrak{g}$  via the inclusion  $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ . This corresponds to the action

$$x.v = x(v) \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

This is then called the *natural representation* of  $\mathfrak{g}$ .

2. If  $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$  is a Lie subalgebra then  $\mathfrak{g}$  acts on  $k^n$  via

$$x.v = x \cdot v \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in k^n,$$

which corresponds to the Lie algebra homomorphism  $\mathfrak{g} \rightarrow \mathfrak{gl}(k^n), x \mapsto (v \mapsto x.v)$ . This is then called the *natural representation* of  $\mathfrak{g}$ .

3. Let  $\mathfrak{g} := \mathfrak{sl}_2(k)$  for any field  $k$ . Then  $k[x, y]$  becomes a representation of  $\mathfrak{g}$  via the homomorphism of Lie algebras  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(k[x, y])$  given by

$$\rho(e) = y \frac{d}{dx}, \quad \rho(h) = y \frac{d}{dy} - x \frac{d}{dx}, \quad \rho(f) = x \frac{d}{dy},$$

where  $x$  and  $y$  also denote the multiplication with the respective variable and  $(e, h, f)$  denotes the standard basis of  $\mathfrak{sl}_2(k)$ . To see that this is a homomorphism of representations notice that for all  $n, m \geq 0$

$$\begin{aligned} e.(x^n y^m) &= nx^{n-1}y^{m+1}, \\ h.(x^n y^m) &= (m - n)x^n y^m, \\ f.(x^n y^m) &= mx^{n+1}y^{m-1}, \end{aligned}$$



where we write  $x^\nu = 0$  and  $y^\mu = 0$  for every  $\nu, \mu < 0$ . From this it follows that for all  $n, m \geq 0$

$$\begin{aligned} e.f.(x^n y^m) - f.e.(x^n y^m) &= (n+1)m x^n y^m - n(m+1)x^n y^m \\ &= (m-n)x^n y^m = h.m = [e, f].(x^n y^m) \end{aligned}$$

as well as

$$\begin{aligned} h.e.(x^n y^m) - e.h.(x^n y^m) &= n(m-n+2)x^{n-1}y^{m+1} - n(m-n)x^{n-1}y^{m+1} \\ &= 2x^{n-1}y^{m+1} = 2e.(x^n y^m) = [h, e].(x^n y^m) \end{aligned}$$

and

$$\begin{aligned} h.f.(x^n y^m) - f.h.(x^n y^m) &= m(m-n-2)x^{n+1}y^{m-1} - m(m-n)x^{n+1}y^{m-1} \\ &= -2x^{n+1}y^{m-1} = -2f.(x^n y^{m-1}) = [h, f].(x^n y^{m-1}). \end{aligned}$$

4. Let  $\mathfrak{g} := \mathfrak{sl}_2(k)$  for any field  $k$ . Then the polynomial ring in one variable  $k[x]$  is a representation of  $\mathfrak{g}$  via the action the homomorphism of Lie algebras  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(k[x])$  with

$$\rho(e) = \frac{d}{dx}, \quad \rho(h) = -2x \frac{d}{dx}, \quad \rho(f) = -\frac{d}{dx}.$$

Then  $\mathfrak{g}$  acts on  $k[x]$  via

$$e.x^n = nx^{n-1}, \quad h.x^n = -2nx^n, \quad f.x^n = nx^{n+1} \quad \text{for every } n \geq 0,$$

where we set  $x^m := 0$  for  $m < 0$ . To see that this is really a representation of  $\mathfrak{g}$  notice that for every  $n \geq 0$

$$e.f.x^n - f.e.x^n = -n(n+1)x^n + n(n-1)x^n = -2nx^n = h.x^n = [e, f].x^n$$

as well as

$$h.e.x^n - e.h.x^n = -2n(n-1)x^{n-1} + 2n^2x^{n-1} = 2nx^{n-1} = 2e.x^n = [h, e].x^n$$

and

$$h.f.x^n - f.h.x^n = 2n(n+1)x^{n+1} - 2n^2x^{n+1} = 2nx^{n+1} = -2f.x^n = [h, f].x^n.$$

5. If  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is a representation of a Lie algebra  $\mathfrak{g}$  and  $\phi: \mathfrak{g}' \rightarrow \mathfrak{g}$  a homomorphism of Lie algebras then via the composition  $\rho \circ \phi: \mathfrak{g}' \rightarrow \mathfrak{gl}(V)$  the vector space  $V$  becomes a representation of  $\mathfrak{g}'$ . This corresponds to the action

$$x.v = \rho(x).v = \rho(\phi(x))(v) \quad \text{for every } x \in \mathfrak{g}' \text{ and } v \in V.$$

6. The map  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), x \mapsto \text{ad}(x)$  is a homomorphism of Lie algebras and hence a representation of  $\mathfrak{g}$ .

**Definition 1.2.6.** Let  $\mathfrak{g}$  be a Lie algebra. Then

$$\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto [x, \cdot]$$

is called the *adjoint representation* of  $\mathfrak{g}$ .

**Remark 1.2.7.** Together with Lemma 1.1.36 it follows that every Lie algebras  $\mathfrak{g}$  acts on itself by derivations of itself via the adjoint representation. This is where much of the structure of Lie algebras comes from and why the Jacobi identity is of interest (I guess).

**Proposition 1.2.8** (New representations from old ones). *Let  $\mathfrak{g}$  be a Lie algebra over an arbitrary field  $k$ .*

1. *If  $(V_i)_{i \in I}$  is a collection of representations of  $\mathfrak{g}$  then  $\bigoplus_{i \in I} V_i$  is a representation of  $\mathfrak{g}$  via*

$$x \cdot \sum_{i \in I} v_i = \sum_{i \in I} x \cdot v_i$$

*or every  $x \in \mathfrak{g}$  and  $v_i \in V_i$  for all  $i \in I$ , with  $v_i = 0$  for all but finitely many  $i \in I$ .*

2. *If  $V$  and  $W$  are representations of  $\mathfrak{g}$  then  $V \otimes W$  is a representation of  $\mathfrak{g}$  via*

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w) \quad \text{for every } x \in \mathfrak{g}, v \in V \text{ and } w \in W.$$

*More generally: If  $V_1, \dots, V_n$  are representations of  $\mathfrak{g}$  then  $V_1 \otimes \dots \otimes V_n$  is a representation of  $\mathfrak{g}$  via*

$$x \cdot (v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \dots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \otimes \dots \otimes v_n.$$

*for every  $x \in \mathfrak{g}$  and  $v_i \in V_i$  for every  $i = 1, \dots, n$ .*

3. *If  $V, W$  are representations of  $\mathfrak{g}$  then  $\text{Hom}_k(V, W)$  is a representation of  $\mathfrak{g}$  via*

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v) \quad \text{for every } x \in \mathfrak{g}, f \in \text{Hom}(V, W) \text{ and } v \in V.$$

4. *By letting  $\mathfrak{g}$  act trivially on  $k$  the dual  $V^* = \text{Hom}_k(V, k)$  becomes a representation of  $\mathfrak{g}$  in the above way, i.e.*

$$(x \cdot \varphi)(v) = -\varphi(x \cdot v) \quad \text{for every } x \in \mathfrak{g}, \varphi \in V^* \text{ and } v \in V.$$

*Proof.* 1. Let  $x, y \in \mathfrak{g}$  and  $v_i \in V_i$  for every  $i \in I$ . Then

$$x \cdot y \cdot \sum_{i=1}^n v_i - y \cdot x \cdot \sum_{i=1}^n v_i = \sum_{i=1}^n (x \cdot y \cdot v_i - y \cdot x \cdot v_i) = \sum_{i=1}^n [x, y] \cdot v_i = [x, y] \cdot \sum_{i=1}^n v_i.$$

2. Let  $x, y \in \mathfrak{g}$  and  $v_i \in V_i$  for every  $i = 1, \dots, n$ . For all  $i, j, m = 1, \dots, n$  set

$$\tilde{w}_m^{ij} := \begin{cases} x \cdot v_i & \text{if } i = m \neq j, \\ y \cdot v_i & \text{if } i \neq m = j, \\ x \cdot y \cdot v_i & \text{if } i = m = j, \\ v_i & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{w}_m^{ij} := \begin{cases} x \cdot v_i & \text{if } i = m \neq j, \\ y \cdot v_i & \text{if } i \neq m = j, \\ y \cdot x \cdot v_i & \text{if } i = m = j, \\ v_i & \text{otherwise.} \end{cases}$$

In particular  $\tilde{w}_m^{ij} = \hat{w}_m^{ij}$  holds for all  $m = 1, \dots, n$  and  $i \neq j$ . Therefore

$$\begin{aligned}
& x.y.(v_1 \otimes \cdots \otimes v_n) - y.x.(v_1 \otimes \cdots \otimes v_n) \\
&= \sum_{i,j=1}^n \tilde{w}_1^{ij} \otimes \cdots \otimes \tilde{w}_m^{ij} - \sum_{i,j=1}^n \hat{w}_1^{ij} \otimes \cdots \otimes \hat{w}_m^{ij} \\
&= \sum_{i=1}^n (\tilde{w}_1^{ii} \otimes \cdots \otimes \tilde{w}_m^{ii} - \hat{w}_1^{ii} \otimes \cdots \otimes \hat{w}_m^{ii}) \\
&= \sum_{i=1}^n (v_1 \otimes \cdots \otimes (x.y.v_i) \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes (y.x.v_i) \otimes \cdots \otimes v_n) \\
&= \sum_{i=1}^n v_1 \otimes \cdots \otimes v_{i-1} \otimes (x.y.v_i - y.x.v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n \\
&= \sum_{i=1}^n v_1 \otimes \cdots \otimes v_{i-1} \otimes ([x, y].v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n = [x, y].(v_1 \otimes \cdots \otimes v_n).
\end{aligned}$$

3. For all  $x, y \in \mathfrak{g}$ ,  $f \in \text{Hom}(V, W)$  it follows for every  $v \in V$  that

$$\begin{aligned}
(x.y.\varphi)(v) - (y.x.\varphi)(v) &= -(y.\varphi)(x.v) + (x.\varphi)(y.v) = \varphi(y.x.v) - (\varphi(x.y.v)) \\
&= -\varphi(x.y.v - y.x.v) = -\varphi([x, y].v) = ([x, y].\varphi)(v). \quad \square
\end{aligned}$$

### 1.2.2. Homomorphisms of representations

**Definition 1.2.9.** Let  $V$  und  $W$  be representations of a  $k$ -Lie algebra  $\mathfrak{g}$ . A  $k$ -linear map  $f: V \rightarrow W$  is called a *homomorphism of representations* if

$$f(x.v) = x.f(v) \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V.$$

$f$  is an *isomorphism of representations* if it is additionally bijective. If  $\rho_V: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  and  $\rho_W: \mathfrak{g} \rightarrow \mathfrak{gl}(W)$  are the corresponding Lie algebra homomomorphisms then  $f$  is a homomorphism of representations if and only if

$$f \circ \rho_V(x) = \rho_W(x) \circ f \quad \text{for every } x \in X.$$

The linear subspace of  $\text{Hom}_k(V, W)$  consisting of the homomorphisms of representations is denoted by  $\text{Hom}_{\mathfrak{g}}(V, W)$ , and for  $V = W$  by  $\text{End}_{\mathfrak{g}}(V) := \text{Hom}_{\mathfrak{g}}(V, V)$ .

**Examples 1.2.10.** Let  $\mathfrak{g}$  be a Lie algebra over a field  $k$ .

1. If  $V$  is a representation of  $\mathfrak{g}$  then  $\text{id}_V: \mathfrak{g} \rightarrow \mathfrak{g}$  is an automorphism of  $V$  as a representation of  $\mathfrak{g}$ .
2. If  $V_1, V_2$  and  $V_3$  are representations of  $\mathfrak{g}$  and  $f: V_1 \rightarrow V_2$  and  $g: V_2 \rightarrow V_3$  are homomorphisms of representations then the composition  $g \circ f: V_1 \rightarrow V_3$  is also a homomorphism of representations.

**Remark 1.2.11.** For a Lie algebra  $\mathfrak{g}$  over a field  $k$  the representations of  $\mathfrak{g}$  together with the homomorphisms of representations between them form a category.

**Remark 1.2.12.** If  $f: V \rightarrow W$  is an isomorphism of representations of a Lie algebra  $\mathfrak{g}$  then the  $k$ -linear map  $f^{-1}: W \rightarrow V$  is also a homomorphism of representations (and therefore an isomorphism of representations) because

$$f^{-1}(x.v) = f^{-1}(x.f(f^{-1}(v))) = f^{-1}(f(x.f^{-1}(v))) = x.f^{-1}(v)$$

for every  $x \in \mathfrak{g}$  and  $v \in V$ . It also follows for every  $x \in \mathfrak{g}$  from  $f\rho_V(x) = \rho_W(x)f$  that  $\rho_V(x)f^{-1} = f^{-1}\rho_W(x)$ .

**Remark 1.2.13.** Given two representations  $V$  and  $W$  of a Lie algebra  $\mathfrak{g}$  a linear map  $f: V \rightarrow W$  is a homomorphism of representations if and only if

$$x.f(v) - f(x.v) = 0 \quad \text{for every } x \in \mathfrak{g} \text{ and } v \in V,$$

which is equivalent to  $x.f = 0$  for every  $x \in \mathfrak{g}$  with respect to the induced action of  $\mathfrak{g}$  on  $\text{Hom}_k(V, W)$ . Hence the homomorphisms of representations are precisely the “invariant” linear maps under the action of  $\mathfrak{g}$ .

**Proposition 1.2.14.** *Let  $\mathfrak{g}$  be a Lie algebra.*

1. *If  $V$  and  $W$  are finite dimensional representations of  $\mathfrak{g}$  then the map*

$$\Phi_1: V^* \otimes W \rightarrow \text{Hom}_k(V, W), \quad \varphi \otimes w \mapsto (v \mapsto \varphi(v)w)$$

*is a homomorphism of representations. If  $V$  and  $W$  are both finite dimensional this is an isomorphism of vector spaces and thus already an isomorphism of representations.*

2. *If  $V_1, \dots, V_r$  and  $W_1, \dots, W_s$  are representations of  $\mathfrak{g}$  then the isomorphism of vector spaces*

$$\begin{aligned} \Phi_2: (V_1 \otimes \cdots \otimes V_r) \otimes (W_1 \otimes \cdots \otimes W_s) &\longrightarrow V_1 \otimes \cdots \otimes V_r \otimes W_1 \otimes \cdots \otimes W_s, \\ (v_1 \otimes \cdots \otimes v_r) \otimes (w_1 \otimes \cdots \otimes w_s) &\longmapsto v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s \end{aligned}$$

*is already an isomorphism of representations.*

3. *If  $V$  and  $W$  are representations of  $\mathfrak{g}$  then the isomorphism of vector spaces*

$$\Phi_3: V \otimes W \rightarrow W \otimes V, \quad v \otimes w \mapsto w \otimes v$$

*is already an isomorphism of representations.*

4. *If  $V_1, V_2$  and  $W$  are representations of  $\mathfrak{g}$  then the isomorphism of vector spaces*

$$\begin{aligned} \Phi_4: (V_1 \otimes V_2) \otimes W &\rightarrow (V_1 \otimes W) \oplus (V_2 \otimes W), \\ (v_1 + v_2) \otimes w &\mapsto (v_1 \otimes w) + (v_2 \otimes w) \end{aligned}$$

*is already an isomorphism of representations.*

5. If  $V_1, \dots, V_n$  are representations of  $\mathfrak{g}$  and  $\sigma \in S_n$  then the isomorphism of vector spaces

$$\Phi_5: V_1 \otimes \cdots \otimes V_n \rightarrow V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}, \quad v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

is already an isomorphism of representations.

6. If  $V_1, \dots, V_n$  and  $W_1, \dots, W_n$  are representations of  $\mathfrak{g}$  and  $f_i: V_i \rightarrow W_i$  for every  $i = 1, \dots, n$  a homomorphism of representations it follows that the map

$$\Phi_6: f_1 \otimes \cdots \otimes f_n: \bigotimes_{i=1}^n V_i \rightarrow \bigotimes_{i=1}^n W_i, \quad v_1 \otimes \cdots \otimes v_n \mapsto f(v_1) \otimes \cdots \otimes f(v_n)$$

is also a homomorphism of representations.

*Proof.* 1. For  $x \in \mathfrak{g}$ ,  $\varphi \in V^*$  and  $w \in W$  it follows for every  $v \in V$  that

$$\begin{aligned} \Phi_1(x.(\varphi \otimes w))(v) &= \Phi_1((x.\varphi) \otimes w + \varphi \otimes (x.w))(v) = (x.\varphi)(v)w + \varphi(v)x.w \\ &= \varphi(v)x.w - \varphi(x.v)w = x.(\varphi(v)w) - \varphi(x.v)w \\ &= x.\Phi_1(\varphi \otimes w)(v) - \Phi_1(\varphi \otimes w)(x.v) = (x.\Phi_1(\varphi \otimes w))(v). \end{aligned}$$

The other statements follow from similar calculations.  $\square$

### 1.2.3. Subrepresentations and irreducible representations

**Definition 1.2.15.** Let  $\mathfrak{g}$  be a Lie algebra and  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  a representation of  $\mathfrak{g}$ . A *subrepresentation* of  $V$  is a linear subspace  $U \subseteq V$  such that  $x.u \in U$  for every  $x \in \mathfrak{g}$  and  $u \in U$ . Equivalently  $U$  is  $\rho(x)$  invariant for every  $x \in \mathfrak{g}$ .

If  $(U_i)_{i \in I}$  is a collection of subrepresentations of  $\mathfrak{g}$  then  $V$  is called the *direct sum* of the  $U_i$  if  $V = \bigoplus_{i \in I} U_i$  as vector spaces.

**Examples 1.2.16.** Let  $\mathfrak{g}$  be a Lie algebra.

1. If  $V$  is a representation of  $\mathfrak{g}$  then  $0$  and  $V$  itself are subrepresentations. These are also called the *trivial subrepresentations*.
2. If  $V$  is a representation and  $(U_i)_{i \in I}$  a collection of subrepresentations  $U_i \subseteq V_i$  then  $\sum_{i \in I} U_i$  is also a subrepresentation of  $V$ .
3. Let  $V$  and  $W$  be representations of a Lie algebra  $\mathfrak{g}$  and  $\varphi: V \rightarrow W$  a homomorphism of representations. Then  $\ker \varphi \subseteq V$  and  $\text{im } \varphi \subseteq W$  are subrepresentations.
4. The subrepresentations of the adjoint representation of  $\mathfrak{g}$  are precisely the ideals in  $\mathfrak{g}$ .

**Definition 1.2.17.** A representation  $V$  of a Lie algebra  $\mathfrak{g}$  is called *irreducible* or *simple* if it has precisely two subrepresentations. Equivalently  $V$  is nonzero and has only the trivial subrepresentations.

The representation  $V$  is called *decomposable* if there exist non-trivial subrepresentations  $U_1, U_2 \subseteq V$  with  $V = U_1 \oplus U_2$ . Otherwise  $V$  is called *indecomposable*.

The representation  $V$  is called *completely decomposable* or *semisimple* if there exists a decomposition  $V = \bigoplus_{i \in I} U_i$  into irreducible subrepresentations  $U_i \subseteq V$ .

**Remark 1.2.18.** By definition every irreducible representation is also indecomposable. The converse does not hold. Irreducible representations are precisely the ones which are both indecomposable and completely reducible.

**Example 1.2.19.** 1. Every one-dimensional representation is irreducible.

2. The adjoint representation of a Lie algebra  $\mathfrak{g}$  is irreducible if and only if  $\mathfrak{g} \neq 0$  and  $\mathfrak{g}$  has no ideals besides 0 and  $\mathfrak{g}$  itself. So  $\mathfrak{g}$  is either the one-dimensional abelian Lie algebra or a simple Lie algebra.

**Lemma 1.2.20 (Schur).** *Let  $\mathfrak{g}$  be a Lie algebra over a field  $k$ .*

1. *Let  $V$  and  $W$  be irreducible representations of  $\mathfrak{g}$  and  $\varphi: V \rightarrow W$  a homomorphism of representation. Then either  $\varphi = 0$  or  $\varphi$  is an isomorphism of representations. In particular  $\text{Hom}_{\mathfrak{g}}(V, W) = 0$  if  $V \not\cong W$ , and  $\text{End}_{\mathfrak{g}}(V)$  is a skew field.*
2. *If  $k$  is algebraically closed and  $V$  a finite-dimensional irreducible representation of  $\mathfrak{g}$  then every  $\varphi \in \text{End}_{\mathfrak{g}}(V)$  is given by multiplication with some  $\lambda \in k$ . In particular  $\text{End}_{\mathfrak{g}}(V) \cong k$  as rings.*

*Proof.* 1. From  $V \neq 0$  and  $W \neq 0$  it follows that  $\varphi$  cannot be zero and an isomorphism at the same time. Suppose that  $\varphi \neq 0$ . Then  $\text{im } \varphi$  is a nonzero subrepresentation of  $W$ , hence  $\text{im } \varphi = W$  because  $W$  is irreducible. Similarly it follows that  $\ker \varphi$  is a proper subrepresentation of  $V$ , hence  $\ker \varphi = 0$  because  $V$  is irreducible. By combining these two observations it follows  $\varphi$  is an isomorphism.

2. As  $k$  is algebraically closed it follows that  $\varphi$  has an eigenvalue  $\lambda \in k$ . Therefore  $\varphi - \lambda$  is an endomorphism of  $V$  as a representation of  $\mathfrak{g}$  with nonzero kernel. Hence it is no isomorphism, so  $\varphi - \lambda \text{id}_V = 0$  because  $\text{End}_{\mathfrak{g}}(V)$  is a skew field.  $\square$

## 1.3. Nilpotent and solvable Lie algebras

### 1.3.1. Definition, examples and properties

**Definition 1.3.1.** Let  $A$  be an associative  $k$ -algebra. An element  $a \in A$  is called *nilpotent* if  $a^n = 0$  for some  $n \geq 1$ . Given a Lie algebra  $\mathfrak{g}$  an element  $x \in \mathfrak{g}$  is called *ad-nilpotent* if  $\text{ad}(x) \in \text{End}_k(\mathfrak{g})$  is nilpotent.

**Lemma 1.3.2.** *If  $A$  is an associative  $k$ -algebra and  $x \in A$  nilpotent then  $x$  is also ad-nilpotent.*

*Proof.* Let  $\lambda_x: A \rightarrow A, a \mapsto xa$  and  $\rho_x: A \rightarrow A, a \mapsto ax$ . Because  $x$  is nilpotent both  $\lambda_x$  and  $\rho_x$  are nilpotent. Because  $A$  is associative  $\lambda_x$  and  $\rho_x$  commute. Hence  $\text{ad}(x) = \lambda_x - \rho_x$  is the sum of two commuting, nilpotent endomorphisms, and therefore also nilpotent.  $\square$

**Definition 1.3.3.** Let  $\mathfrak{g}$  be a Lie algebra. Define  $\mathfrak{g}^0 := \mathfrak{g}$  and  $\mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i]$  for all  $i \in \mathbb{N}$ . Then

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

is called the *central series* of  $\mathfrak{g}$ . Also define  $\mathfrak{g}^{(0)} := \mathfrak{g}$  and  $\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$  for all  $i \in \mathbb{N}$ . Then

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$$

is called the *derived series* of  $\mathfrak{g}$ .  $\mathfrak{g}$  is called *nilpotent* if  $\mathfrak{g}^i = 0$  for some  $i$  and *solvable* if  $\mathfrak{g}^{(i)} = 0$  for some  $i$ .

**Examples 1.3.4.** 1. Every nilpotent Lie algebra  $\mathfrak{g}$  is also solvable because  $\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^i$  for every  $i \in \mathbb{N}$ .

2. The upper triangular matrices  $\mathfrak{t}_n(k)$  are solvable. But they are not nilpotent.
3. The strictly upper triangular matrices  $\mathfrak{n}_n(k)$  not only solvable but also nilpotent.
4. If  $n \geq 2$  then  $\mathfrak{sl}_2(\mathbb{C})$  is simple and therefore  $[\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$ . Since  $[\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$  it follows that  $\mathfrak{gl}_n(\mathbb{C})$  is not solvable.
5. If  $\mathfrak{g}$  is abelian then  $\mathfrak{g}$  is nilpotent and therefore also solvable.
6. Every one-dimensional Lie algebra is abelian and therefore nilpotent and also solvable. The same goes for the two-dimensional abelian Lie algebra. The two-dimensional non-abelian Lie algebra  $\mathfrak{g}$  has a basis  $x, y$  with  $[x, y] = x$ . Therefore  $\mathfrak{g}$  is solvable but not nilpotent.
7. A *Heisenberg Lie algebra* consists of a real vector space with basis  $P_1, \dots, P_n, Q_1, \dots, Q_n, C$  together with the Lie bracket satisfying the following conditions:

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_i, C] = 0 \quad \text{and} \quad [P_i, Q_j] = \delta_{ij}C.$$

This defines a nilpotent Lie algebra.

**Proposition 1.3.5.** Let  $\mathfrak{g}$  be a Lie algebra.

1. If  $\mathfrak{h}$  is a Lie algebra and  $f: \mathfrak{g} \rightarrow \mathfrak{h}$  a Lie algebras homomorphism then  $f(\mathfrak{g})^i = f(\mathfrak{g}^i)$  and  $f(\mathfrak{g})^{(i)} = f(\mathfrak{g}^{(i)})$  for all  $i \geq 0$ .
2. If  $\mathfrak{g}$  is nilpotent (resp. solvable) then any Lie subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  and any quotient of  $\mathfrak{g}$  (by an ideal  $I$ ) is nilpotent (resp. nilpotent).
3. If  $I \trianglelefteq \mathfrak{g}$  with  $I \subseteq Z(\mathfrak{g})$  and  $\mathfrak{g}/I$  is nilpotent then  $\mathfrak{g}$  is nilpotent.
4. If  $\mathfrak{g} \neq 0$  is nilpotent then  $Z(\mathfrak{g}) \neq 0$ .
5. If  $\mathfrak{g}$  is nilpotent and  $x \in \mathfrak{g}$  then  $x$  is ad-nilpotent.
6. If  $I \trianglelefteq \mathfrak{g}$  then  $I^i$  and  $I^{(i)}$  are ideals inside  $\mathfrak{g}$  for all  $i \geq 0$ .

*Proof.* 1. It suffices to show that for any two subsets  $X, Y \subseteq \mathfrak{g}$

$$f([X, Y]) = [f(X), f(Y)]$$

the statement then follows inductively. It holds because  $f$  is a Lie algebra homomorphism and therefore

$$\begin{aligned} f([X, Y]) &= f(\text{span}_k\{[x, y] \mid x \in X, y \in Y\}) \\ &= \text{span}_k\{f([x, y]) \mid x \in X, y \in Y\} \\ &= \text{span}_k\{[f(x), f(y)] \mid x \in X, y \in Y\} \\ &= \text{span}_k\{[x', y'] \mid x' \in f(X), y' \in f(Y)\} \\ &= [f(X), f(Y)]. \end{aligned}$$

2. The statement about subalgebras follows from  $\mathfrak{h}^i \subseteq \mathfrak{g}^i$  and  $\mathfrak{h}^{(i)} \subseteq \mathfrak{g}^{(i)}$  for all  $i \in \mathbb{N}$ . The statement about quotient follows by using the canonical projection  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I$ . Because  $\pi$  is a Lie algebra homomorphism it follows that

$$(\mathfrak{g}/I)^i = \pi(\mathfrak{g}^i) = \pi(\mathfrak{g}^i) = 0$$

for  $i$  big enough. For solvable  $\mathfrak{g}$  the corresponding statements follow in the same way.

3. Let  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I$  be the canonical projection. Because  $\mathfrak{g}/I$  is nilpotent there exists some  $i \geq 0$  with  $(\mathfrak{g}/I)^i = 0$  and therefore

$$0 = (\mathfrak{g}/I)^i = \pi(\mathfrak{g}^i) = \pi(\mathfrak{g}^i).$$

Thus  $\mathfrak{g}^i \subseteq I \subseteq Z(G)$  and hence  $\mathfrak{g}^{i+1} = 0$ .

4. Let  $i \in \mathbb{N}$  be minimal with  $\mathfrak{g}^i \neq 0$  but  $\mathfrak{g}^{i+1} = 0$ . Then  $\mathfrak{g}^i \subseteq Z(\mathfrak{g})$  and thus  $Z(\mathfrak{g}) \neq 0$ .

5. Since  $\mathfrak{g}$  is nilpotent there exists some  $i \in \mathbb{N}$  with  $\mathfrak{g}^i = 0$ . Then

$$(\text{ad}(x))^i(\mathfrak{g}) \subseteq \mathfrak{g}^i = 0,$$

so  $(\text{ad}(x))^i = 0$ .

6. This follows inductively by using that  $[I, J]$  is an ideal inside  $\mathfrak{g}$  for any  $I, J \trianglelefteq \mathfrak{g}$ .  $\square$

**Corollary 1.3.6.** *If  $I \trianglelefteq \mathfrak{g}$  is an ideal inside a Lie algebra  $\mathfrak{g}$  then  $\mathfrak{g}$  is solvable if and only if both  $I$  and  $\mathfrak{g}/I$  are solvable.*

*Proof.* If  $\mathfrak{g}$  is solvable then  $I$  and  $\mathfrak{g}/I$  are also solvable by Proposition 1.3.5. Suppose that on the other hand both  $I$  and  $\mathfrak{g}/I$  are solvable. Then there exists  $i_1, i_2 \in \mathbb{N}$  with  $(\mathfrak{g}/I)^{(i_1)} = 0$  and  $I^{i_2} = 0$ . Let  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I, x \mapsto x + I$  be the canonical projection. Because

$$0 = (\mathfrak{g}/I)^{(i_1)} = \pi(\mathfrak{g}^{(i_1)}) = \pi(\mathfrak{g}^{(i_1)})$$



it follows that  $\mathfrak{g}^{(i_1)} \subseteq \ker \pi = I$ . Thus

$$\mathfrak{g}^{(i_1+i_2)} = (\mathfrak{g}^{(i_1)})^{i_2} \subseteq I^{i_2} = 0,$$

which shows that  $\mathfrak{g}$  is solvable.  $\square$

**Remark 1.3.7.** The analogous statement about nilpotency does not necessarily hold. Take for example the two-dimensional non-abelian Lie algebra  $\mathfrak{g}$ , which has a basis  $x, y$  with  $[x, y] = x$ . Then the one-dimensional linear subspace  $I := kx$  is an abelian ideal in  $\mathfrak{g}$  and in particular nilpotent. The quotient  $\mathfrak{g}/I$  is one-dimensional and therefore also nilpotent. But  $\mathfrak{g}$  itself is not nilpotent.

**Corollary 1.3.8.** *Let  $\mathfrak{g}$  be a Lie algebra and  $I, J \trianglelefteq \mathfrak{g}$  two solvable ideal. Then  $I + J$  is also solvable.*

*Proof.* Because  $J$  is solvable the same goes for  $J/(I \cap J)$ . Hence in the short exact sequence

$$0 \rightarrow I \rightarrow I + J \rightarrow (I + J)/I \rightarrow 0$$

both  $I$  and  $(I+J)/I \cong J/(I \cap J)$  are solvable. Hence  $I+J$  is solvable by Corollary 1.3.6.  $\square$

**Definition 1.3.9.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. It follows from Corollary 1.3.8 that  $\mathfrak{g}$  contains a unique maximal solvable ideal. This ideal is called the *radical* of  $\mathfrak{g}$  and is denoted by  $\text{rad } \mathfrak{g}$ .

**Remark 1.3.10.** If  $\mathfrak{g}$  is a Lie algebra and  $I, J \trianglelefteq \mathfrak{g}$  two nilpotent ideals, then it can be shown that the ideal  $I + J$  is also a nilpotent. It follows that every finite dimensional Lie algebra  $\mathfrak{g}$  has a unique maximal nilpotent ideal, which is then called the *nilradical* of  $\mathfrak{g}$ .

### 1.3.2. Engel's theorem

From now on *all* fields over which we work will be assumed to be algebraically closed, unless otherwise specified.

If  $V$  is an  $n$ -dimensional vector space over  $k$  and  $x \in \text{End}_k(V)$  a nilpotent endomorphism then 0 is the only eigenvalue of  $x$  (and occurs with multiplicity  $n$ ) (here it is used that  $k$  is algebraically closed). Hence there exists an eigenvector  $v \in V$ ,  $v \neq 0$  with  $x(v) = 0$ . The following proposition generalizes this observations for linear Lie algebras consisting of nilpotent endomorphisms.

**Proposition 1.3.11.** *Let  $V \neq 0$  be a finite dimensional vector space and  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  a Lie subalgebra such that every  $x \in \mathfrak{g}$  is nilpotent. Then there exists  $v \in V$  with  $v \neq 0$  and  $x(v) = 0$  for every  $x \in \mathfrak{g}$ , i.e.  $v$  is a common eigenvector of all  $x \in \mathfrak{g}$  (all of which are nilpotent and thus have 0 as their only eigenvalue).*

*Proof.* The statement can be then shown by induction over  $\dim \mathfrak{g}$ . For  $\dim \mathfrak{g} = 0$  the statement follows from  $\mathfrak{g} = 0$  and for  $\dim \mathfrak{g} = 1$  the statement follows as previously discussed from  $\mathfrak{g} = kx$  with  $x$  being a nilpotent endomorphism of  $V$ .

So let  $\dim \mathfrak{g} \geq 2$  and suppose that the statement holds for all smaller dimensions. Let  $I \subseteq \mathfrak{g}$  be a maximal proper Lie subalgebra (such a subalgebra exists because it is precisely one of maximal dimension strictly smaller than  $n$ ). Any  $x \in \mathfrak{g}$  with  $x \neq 0$  spans a one-dimensional subalgebra  $kx$  of  $\mathfrak{g}$ ; because  $\dim \mathfrak{g} \geq 2$  it is a proper one. This shows that  $I \neq 0$ . It turns out that  $I$  is already an ideal in  $\mathfrak{g}$ :

By assumption  $\mathfrak{g}$  consists of nilpotent endomorphisms and therefore of ad-nilpotent elements. In particular every  $x \in I$  acts nilpotently on  $\mathfrak{g}$  via  $\text{ad}(x)$  with  $I$  being an  $\text{ad}(x)$ -invariant linear subspace. Therefore every  $x \in I$  acts on the quotient vector space  $\mathfrak{g}/I$  by an induced nilpotent endomorphism

$$\overline{\text{ad}}(x): \mathfrak{g}/I \rightarrow \mathfrak{g}/I, \quad y + I \mapsto \text{ad}(x)(y) + I = [x, y] + I.$$

As the map  $\overline{\text{ad}}: I \rightarrow \mathfrak{gl}(\mathfrak{g}/I)$  is a homomorphism of Lie algebras (because  $\text{ad}$  is) the image  $\{\overline{\text{ad}}(x) \mid x \in I\} \subseteq \mathfrak{gl}(\mathfrak{g}/I)$  is a Lie subalgebra, consisting of nilpotent endomorphisms. From  $I \neq 0$  it follows that  $\dim \mathfrak{g}/I < \dim \mathfrak{g}$ , so by induction assumption there exists some  $y \in \mathfrak{g}$  with  $\overline{\text{ad}}(x)(y + I) = 0$  for every  $x \in I$  and  $y + I \neq 0 + I$ . Hence  $[x, y] \in I$  for every  $x \in I$  but  $y \notin I$ . Hence  $y \in N_{\mathfrak{g}}(I)$  with  $y \notin I$ , so  $I$  is properly contained in its normalizer. As  $I$  is a maximal proper subalgebra of  $\mathfrak{g}$  it follows that  $N_{\mathfrak{g}}(I) = \mathfrak{g}$ , so  $I$  is an ideal. It is even one of codimension 1:

If  $I$  had not codimension 1 then  $\dim \mathfrak{g}/I > 1$ . Then  $\mathfrak{g}/I$  contains a one-dimensional proper subalgebra  $L$  (as seen above), and the preimage  $\pi^{-1}(L)$  under the canonical projection  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I$  is then a proper subalgebra of  $\mathfrak{g}$  properly containing  $I$ , which contradicts the maximality of  $I$ . Hence  $I$  has codimension 1.

As  $I \subseteq \mathfrak{g}$  has codimension 1 there exists some  $y \in \mathfrak{g}$  with  $\mathfrak{g} = I \oplus ky$  (as vector spaces). Because  $\dim I < \dim \mathfrak{g}$  it follows from the induction assumption that

$$U := \{v \in V \mid x(v) = 0 \text{ for every } x \in I\} = \bigcap_{x \in I} \ker x$$

is a nonzero linear subspace of  $V$ . It suffices to show that  $U$  is  $y$ -invariant: Then there exists some eigenvector  $u \in U$  of  $y$  for which necessarily  $y(u) = 0$ . If  $u \in U$  then  $[x, y](u) \in I$  for every  $x \in I$  because  $I \trianglelefteq \mathfrak{g}$  and therefore

$$x(y(u)) = [x, y](u) - y(x(u)) = 0 - y(0) = 0.$$

Hence  $y(u) \in U$ , so  $U$  is  $y$ -invariant.  $\square$

**Proposition 1.3.12.** *Let  $V$  be a finite dimensional vector space with  $n = \dim V$  and  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  a Lie subalgebra. Then the following are equivalent:*

1.  $\mathfrak{g}$  consists of nilpotent endomorphisms.
2. There exists a complete flag of  $V$

$$V = V_n \supseteq V_{n-1} \supseteq V_{n-2} \supseteq \cdots \supseteq V_1 \supseteq V_0 = 0,$$

with  $x(V_i) \subseteq V_{i-1}$  for every  $i = 1, \dots, n$ .

3. There exists a basis of  $V$  with respect to which every  $x \in \mathfrak{g}$  is represented by an strictly upper triangular matrix.

*Proof.* The implication **1**  $\Rightarrow$  **2** can be shown by induction over  $\dim V$ . For  $\dim V = 1$  set  $V_0 := 0$  and  $V_1 := V$ . By assumption every  $x \in \mathfrak{g}$  acts nilpotent on  $V$ , so  $x(V) = 0$  because  $V$  is one-dimensional. Thus  $V = V_1 \supseteq V_0 = 0$  is a complete flag for  $V$  satisfying the conditions.

Now let  $\dim V = n \geq 2$  and suppose the statement holds for all smaller dimensions. Let  $v \in V$ ,  $v \neq 0$  with  $x(v) = 0$  for every  $x \in \mathfrak{g}$  and  $W := V/kv$ . Every  $x \in \mathfrak{g}$  induces an endomorphism

$$\bar{x}: W \rightarrow W, \quad v + kv \mapsto x(v) + kv.$$

By induction assumption exists a complete flag

$$W = W_{n-1} \supseteq W_{n-2} \supseteq W_{n-3} \supseteq \cdots \supseteq W_1 \supseteq W_0 = 0$$

with  $\bar{x}(W_i) \subseteq W_{i-1}$  for every  $x \in \mathfrak{g}$  and  $i = 1, \dots, n-1$ . By setting  $V_i := \pi^{-1}(W_{i-1})$  for every  $i = 1, \dots, n$  and  $V_0 = 0$  it follows that

$$V = V_n \supseteq V_{n-1} \supseteq V_{n-2} \supseteq \cdots \supseteq V_1 \supseteq V_0 = 0,$$

is a complete flag of  $V$ . On the one hand  $x(V_1) = x(kv) = 0 = V_0$  for every  $x \in \mathfrak{g}$  and on the other hand

$$\pi(x(V_i)) = \bar{x}(\pi(V_i)) = \bar{x}(W_{i-1}) \subseteq W_{i-2}$$

and therefore  $x(V_i) \subseteq \pi^{-1}(W_{i-2}) = V_{i-1}$  for every  $i = 2, \dots, n$ .

The implications **2**  $\Rightarrow$  **3** and **3**  $\Rightarrow$  **1** are basic facts from linear algebra.  $\square$

**Theorem 1.3.13** (Engel). *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. Then  $\mathfrak{g}$  is nilpotent if and only if all its elements are ad-nilpotent.*

*Proof.* If  $\mathfrak{g}$  is nilpotent then there exists some  $i \in \mathbb{N}$  with  $\mathfrak{g}^i = 0$ , from which it follows from  $\text{ad}(x)^i(y) \in \mathfrak{g}^i$  for every  $x, y \in \mathfrak{g}$  that  $\text{ad}(x)^i = 0$  for every  $x \in \mathfrak{g}$ , hence every  $x \in \mathfrak{g}$  is ad-nilpotent.

On the other hand suppose that  $\mathfrak{g}$  consists of ad-nilpotent elements. If  $\mathfrak{g} = Z(\mathfrak{g})$  then  $\mathfrak{g}$  is abelian and hence nilpotent, so it suffices to show the statement under the additional assumption that  $Z(\mathfrak{g}) \subsetneq \mathfrak{g}$ . Because  $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{ad } \mathfrak{g}$  is a Lie subalgebra of  $\mathfrak{gl}(\mathfrak{g})$  consisting of nilpotent elements it follows from Proposition 1.3.12 that  $\mathfrak{g}/Z(\mathfrak{g})$  is isomorphic to a Lie subalgebra of  $\mathfrak{n}_n(k)$  for  $n = \dim \mathfrak{g}/Z(\mathfrak{g}) \geq 1$ . Because  $\mathfrak{n}_n(k)$  is nilpotent the same goes for  $\mathfrak{g}/Z(\mathfrak{g})$  as seen in Proposition 1.3.5.  $\square$

**Remark 1.3.14.** It is not true that every nilpotent Lie-subalgebra  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  with  $V$  being a finite dimensional vector space is represented by upper triangular matrices with respect to some basis of  $V$ . For example the onedimensional subalgebra  $k \text{id}_V \subseteq \mathfrak{gl}(V)$  is abelian and hence nilpotent, but with respect to every basis of  $V$  represented by  $kI \subseteq \mathfrak{gl}_n(k)$ .

### 1.3.3. Lie's theorem

From now on we will not only require *every* field  $k$  we work with to be algebraically closed, but also to be of characteristic 0. Unless otherwise stated this holds up to the last page (page 57) of this text. In particular all Lie algebras and vector spaces will be assumed to have such a field as their ground field, even if not explicitly stated.

**Definition 1.3.15.** Let  $V$  be a representation of a Lie algebra  $\mathfrak{g}$ . For  $\lambda \in \mathfrak{g}^*$  the linear subspace

$$V_\lambda := \{v \in V \mid x.v = \lambda x \text{ for every } x \in \mathfrak{g}\}$$

is called the  $\mathfrak{g}$ -weight space of  $V$  with weight  $\lambda$ .

**Lemma 1.3.16** (Invariance Lemma). *Let  $V$  be a finite dimensional representation of a Lie algebra  $\mathfrak{g}$  and  $I \trianglelefteq \mathfrak{g}$  an ideal. Then  $V$  is also a representation of  $I$  by restriction of the action of  $\mathfrak{g}$  on  $V$  to  $I$ . For  $\lambda \in I^*$  let  $V_\lambda$  be the  $I$ -weight space of  $V$  with weight  $\lambda$ . Then  $V_\lambda$  is a subrepresentation of  $\mathfrak{g}$ .*

*Proof.* For  $v \in V$  and  $x_1, \dots, x_n \in \mathfrak{g}$  we will write

$$x_1 \cdots x_n v := x_1.(\dots.(x_n.v)).$$

If  $V_\lambda = 0$  the statement is clear, so for this proof we fix some  $\lambda \in I^*$  with  $V_\lambda \neq 0$ .

That  $V_\lambda$  is a subrepresentation of  $\mathfrak{g}$  means that  $yv \in V_\lambda$  for every  $y \in \mathfrak{g}$  and  $v \in V_\lambda$ , which is equivalent to  $xyv = \lambda(x)yv$  for every  $x \in I$ ,  $y \in \mathfrak{g}$  and  $v \in V_\lambda$ . Because

$$xyv = [x, y]v + yxv = \lambda([x, y])v + \lambda(x)yv \quad \text{for every } x \in I, y \in \mathfrak{g} \text{ and } v \in V_\lambda$$

this is equivalent to  $\lambda([x, y]) = 0$  for every  $x \in I$  and  $y \in \mathfrak{g}$ .

Until further notice we fix some  $y \in \mathfrak{g}$  and  $v \in V_\lambda$  with  $v \neq 0$ . As  $V$  is finite dimensional there exists some maximal  $n \geq 1$  such that  $v, yv, \dots, y^n v$  are linearly independent. Let

$$W_i := \text{span}_k(v, yv, \dots, y^i v) \quad \text{for every } i = 0, \dots, n.$$

Because  $v, yv, \dots, y^n v, y^{n+1}v$  are linearly dependent it follows that  $W_n$  is invariant under the action of  $y$ .

**Claim.** *The linear subspace  $W_i$  is for every  $i = 0, \dots, n$  a subrepresentation of  $I$ . With respect to the basis  $w, yw, \dots, y^i w$  of  $W_i$  the action of  $x \in I$  is represented by an upper triangular matrix where every diagonal entry is  $\lambda(x)$ .*

*Proof.* The claim can be proven by induction over  $i$ . As  $W_0 = kv$  with  $xv = \lambda(x)v$  for every  $x \in I$  the claim holds for  $i = 0$ . Suppose that  $i < n$  and that the claim holds for  $W_0, \dots, W_i$ . If  $x \in I$  then also  $[x, y] \in I$  and therefore

$$xy^{i+1}v = \underbrace{[x, y]y^i v + yxy^i v}_{\substack{\in W_i \\ \text{by induction}}} \equiv yxy^i v \pmod{W_i}.$$

By induction it is not only  $xy^i v \in W_i$  but also  $xy^i v + W_{i-1} = \lambda(x)y^i v + W_{i-1}$ . Therefore

$$yxy^i v \equiv \lambda(x)y^{i+1}v \pmod{W_i}.$$

This shows the claim for  $W_{i+1}$ .  $\square$

Let  $x \in I$ . As  $[x, y] \in I$  it follows from the previous claim that the  $(n+1)$ -dimensional linear subspace  $W_n$  is invariant under the action of  $[x, y]$ , which is given by an endomorphism  $\phi_{[x,y]} \in \text{End}_k(W_n)$ , and that  $\phi_{[x,y]}$  is represented by an upper triangular matrix for which all diagonal entries are  $\lambda([x, y])$ . It follows that in particular

$$\text{tr } \phi_{[x,y]} = (n+1)\lambda([x, y]) \quad (2)$$

On the other hand  $W_n$  is invariant under the action of both  $x$  (by the claim) and  $y$ , which act by endomorphisms  $\phi_x, \phi_y \in \text{End}_k(V)$ . As  $V$  is a representation of the Lie algebra  $\mathfrak{g}$  it follows that  $\phi_{[x,y]} = [\phi_x, \phi_y]$  and thus  $\text{tr } \phi_{[x,y]} = 0$ . Together with (2) it follows that  $\lambda([x, y]) = 0$ .  $\square$

As a generalization of Proposition 1.3.11 we have the following result about solvable linear Lie algebras.

**Theorem 1.3.17** (Lie). *Let  $V \neq 0$  be a finite dimensional  $k$ -vector space and  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  a solvable Lie subalgebra. Then there exists a common eigenvector for  $\mathfrak{g}$ , i.e. some  $v \in V$ ,  $v \neq 0$  with  $x(v) \in kv$  for every  $x \in \mathfrak{g}$ .*

*Proof.* The statement can be shown by induction over  $\dim \mathfrak{g}$ . If  $\dim \mathfrak{g} = 0$  then  $\mathfrak{g} = 0$  and any  $v \in V$  with  $v \neq 0$  does the job. If  $\dim \mathfrak{g} = 1$  then  $\mathfrak{g} = kx$  for some  $x \in \mathfrak{gl}(V)$  with  $x \neq 0$ . Then any eigenvector of  $x$  does the job (since  $k$  is assumed to be algebraically closed and  $V \neq 0$  such an eigenvector does exist).

Suppose that  $\dim \mathfrak{g} = n \geq 2$  and the statement holds for every smaller dimension. Similarly to the proof of Proposition 1.3.11 we will split this proof into four consecutive parts:

1. Finding an ideal  $I \trianglelefteq \mathfrak{g}$  of codimension 1.
2. Finding common eigenvectors for  $I$  by induction.
3. Showing that  $\mathfrak{g}$  stabilizes as nonzero subspace  $U \subseteq V$  of such eigenvectors.
4. Writing  $\mathfrak{g} = I \oplus ky$  (as vector spaces) and finding an eigenvector of  $y$  in  $U$ .

For the first step notice that  $\mathfrak{g}$  is nonzero but solvable, so  $[\mathfrak{g}, \mathfrak{g}] \trianglelefteq \mathfrak{g}$  is a proper ideal. Hence  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  is nonzero abelian Lie algebra. Therefore there exists a linear subspace  $J \subseteq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  of codimension 1 and  $J$  is an ideal in  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . Hence the preimage  $I = \pi^{-1}(J)$  for the canonical projection  $\mathfrak{g} \rightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ ,  $x \mapsto x + [\mathfrak{g}, \mathfrak{g}]$  is an ideal in  $\mathfrak{g}$  of codimension 1.

For the second step notice that because  $\mathfrak{g}$  is solvable the same goes for  $I$ . So by induction hypothesis there exists a common eigenvector for  $I$ . Hence there exists some

$\lambda \in I^*$  with  $U := V_\lambda \neq 0$ , where we view  $V$  as a representation of  $\mathfrak{g}$  via  $x.v = x(v)$  for every  $x \in \mathfrak{g}$  and  $v \in V$ .

The third step follows directly from the invariance lemma.

For the fourth step let  $y \in \mathfrak{g}$  with  $\mathfrak{g} = I \oplus ky$  (as vector spaces). Since  $\mathfrak{g}$  stabilizes  $U$  this holds in particular for  $y$ . As  $U \neq 0$  it follows that there exists some eigenvector of  $y$  inside of  $U$ , which is then a common eigenvector for  $\mathfrak{g}$ .  $\square$

**Remark 1.3.18.** The proof for Lie's theorem given in the lecture is basically a less structured version of the one in [Hum72, §4.1], from where we took the idea of breaking down the proof into four steps to emphasize the similarities with the proof of Proposition 1.3.11 (which we found very useful for understanding the structure of the previous proof).

**Proposition 1.3.19.** *Let  $V \neq 0$  be a finite dimensional  $k$ -vector space with  $n = \dim V$  and  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  a Lie subalgebra. Then the following are equivalent:*

1.  $\mathfrak{g}$  is solvable.
2.  $\mathfrak{g}$  stabilizes some complete flag of  $V$ , i.e. there exists a complete flag

$$V = V_n \supseteq V_{n-1} \supseteq V_{n-2} \supseteq \cdots \supseteq V_1 \supseteq V_0 = 0,$$

with  $x(V_i) \subseteq V_i$  for every  $x \in \mathfrak{g}$  and  $i = 0, \dots, n$ .

3. There exists a basis of  $V$  with respect to which every  $x \in \mathfrak{g}$  is represented by an upper triangular matrix. In particular  $\mathfrak{g}$  is isomorphic to a Lie-subalgebra of  $\mathfrak{t}_n(k)$  for  $n = \dim V$ .

**Corollary 1.3.20.** *A finite-dimensional  $k$ -Lie algebra  $\mathfrak{g}$  is solvable if and only if  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent.*

*Proof.* If  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent then  $[\mathfrak{g}, \mathfrak{g}]^{(i)} = 0$  for some  $i \in \mathbb{N}$ . Hence  $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}]^{(i)} = 0$ , so  $\mathfrak{g}$  is solvable.

Suppose that  $\mathfrak{g}$  is solvable. Then  $\text{ad } \mathfrak{g} \cong \mathfrak{g}/Z(\mathfrak{g})$  is a solvable subalgebra von  $\mathfrak{gl}(\mathfrak{g})$ . By Lie's theorem there exists a basis of  $\mathfrak{g}$  with respect to which  $\text{ad } x$  is represented by an upper triangular matrix for each  $x \in \mathfrak{g}$ . As  $\text{ad}$  is an homomorphism of Lie algebras it follows that with respect to this basis  $\text{ad}(x)$  is represented by a strictly upper triangular matrix for every  $x \in [\mathfrak{g}, \mathfrak{g}]$ . Hence every  $x \in [\mathfrak{g}, \mathfrak{g}]$  is  $\text{ad}$ -nilpotent, and therefore also  $\text{ad}_{[\mathfrak{g}, \mathfrak{g}]}$ -nilpotent. By Engel's theorem  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent.  $\square$

**Corollary 1.3.21.** *Every irreducible representation of a solvable Lie algebra  $\mathfrak{g}$  over  $k$  is onedimensional.*

*Proof.* Let  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be an irreducible representation of  $\mathfrak{g}$ , and therefore in particular  $V \neq 0$ . Then  $\text{im } \rho \subseteq \mathfrak{gl}(V)$  is also solvable and by Lie's theorem there exists a common eigenvector  $v \in V$ ,  $v \neq 0$  for  $\text{im } \rho$ . Because  $x.v = \rho(x)(v) \in kv$  for every  $x \in \mathfrak{g}$  it follows that the onedimensional linear subspace  $kv \subseteq V$  is a nonzero subrepresentation of  $V$ . Because  $V$  is irreducible it follows that  $V = kv$ .  $\square$

**Remark 1.3.22.** Corollary 1.3.21 is actually equivalent to Lie's theorem: If  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  is a Lie subalgebra with then  $V$  is a representation of  $\mathfrak{g}$  via  $x.v = x(v)$  for every  $x \in \mathfrak{g}$  and  $v \in V$ . If  $V \neq 0$  is finite dimensional then  $V$  contains an irreducible subrepresentation  $U$  of  $\mathfrak{g}$  (simply take some nonzero subrepresentation of minimal dimension.) If  $\mathfrak{g}$  additionally is solvable then by Corollary 1.3.21 the irreducible subrepresentation  $U$  is onedimensional, hence of the form  $U = kv$  for some  $v \in V$  with  $v \neq 0$ . From the definition of the action of  $\mathfrak{g}$  on  $V$  it follows that  $v$  is common eigenvector of  $\mathfrak{g}$ .

As a consequence of this Lie's theorem as formulated in Theorem 1.3.17 was called "Lie's theorem – concrete form" in the lecture while Corollary 1.3.21 was stated as "Lie's theorem – abstract version".

**Remark 1.3.23.** Corollary 1.3.21 does not hold for general fields  $k$ , even if algebraically closed. To see this let  $k$  be an algebraically closed field with  $\text{char } k = 2$  and  $\mathfrak{g} := \mathfrak{sl}_2(k)$ . Then  $\mathfrak{g}$  has a basis  $(e, h, f)$  where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with

$$[h, e] = [h, f] = 0 \quad \text{and} \quad [e, f] = h.$$

Hence  $\mathfrak{g}$  is solvable. Let  $V := k^2$  be the natural representation of  $\mathfrak{g}$ , i.e.  $\mathfrak{g}$  acts on  $V$  by  $x.v = x(v)$  for every  $x \in \mathfrak{g}$  and  $v \in V$ . Then

$$e \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{and} \quad f \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix} \quad \text{for every} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in V.$$

It follows that if  $U \subseteq V$  is a nonzero subrepresentation then  $U$  contains either  $e_1$  or  $e_2$ , and therefore also the other one. Hence  $U = V$ , which shows that  $V$  is an irreducible representation of  $\mathfrak{g}$ .

## 1.4. The Killing form and Cartan's criterion

### 1.4.1. Associative bilinear forms and the Killing form

**Definition 1.4.1.** Let  $\mathfrak{g}$  be a Lie algebra over an arbitrary field  $k$ . A bilinear form  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  is called *associative* if

$$\beta(x, [y, z]) = \beta([x, y], z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

**Remark 1.4.2.** As  $\mathfrak{g}$  acts on itself by the adjoint representation it also acts on  $(\mathfrak{g} \otimes_k \mathfrak{g})^*$  as described in Proposition 1.2.8, i.e. for every  $\beta \in (\mathfrak{g} \otimes_k \mathfrak{g})^*$  and all  $x, y, z \in \mathfrak{g}$

$$\begin{aligned} (y \cdot \beta)(x \otimes z) &= -\beta(y \cdot (x \otimes z)) = -\beta((y \cdot x) \otimes z + x \otimes (y \cdot z)) \\ &= -\beta([y, x] \otimes z) - \beta(x \otimes [y, z]) = \beta([x, y] \otimes z) - \beta(x \otimes [y, z]). \end{aligned}$$

Identifying  $(\mathfrak{g} \otimes_k \mathfrak{g})^*$  with the bilinear forms on  $\mathfrak{g}$  via the universal property of the tensor product it follows that a bilinear form  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  is associative if and only if

$$y.\beta = 0 \quad \text{for every } y \in \mathfrak{g}.$$

Because of this associative bilinear forms on  $\mathfrak{g}$  are also called *invariant*.

**Lemma 1.4.3.** *Let  $\mathfrak{g}$  be a Lie algebra and  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  a symmetric bilinear form. Then  $\beta$  is associative if and only if the map*

$$\varphi: \mathfrak{g} \rightarrow \mathfrak{g}^*, \quad x \mapsto \beta(x, \cdot)$$

*is a homomorphism of representations (where  $\mathfrak{g}$  acts on itself by the adjoint representation and therefore on  $\mathfrak{g}^*$  as described in Proposition 1.2.8).*

*Proof.* For every  $x, y, z \in \mathfrak{g}$  and  $y \in \mathfrak{g}$  the equalities

$$(x.\varphi(y))(z) = -\varphi(y)(x.z) = -\beta(y, x.z) = -\beta(y, [x, z])$$

and

$$\varphi(x.y)(z) = \beta(x.y, z) = \beta([x, y], z) = -\beta([y, x], z)$$

hold. Hence the associativity of  $\beta$  is equivalent to the equality  $x.\varphi(y) = \varphi(x.y)$  holding for all  $x, y \in \mathfrak{g}$ .  $\square$

**Corollary 1.4.4.** *If  $\mathfrak{g}$  is a finite-dimensional Lie algebra and  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  an associative and symmetric bilinear form which is non-degenerate then the map*

$$\varphi: \mathfrak{g} \rightarrow \mathfrak{g}^*, \quad x \mapsto \varphi(x, \cdot)$$

*is an isomorphism of representations.*

**Definition 1.4.5.** Let  $V$  be a vector space and  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  a symmetric bilinear form. Then

$$\text{rad } \beta := \{x \in V \mid \beta(x, y) = 0 \text{ for every } y \in V\}$$

is the *radical* of  $\beta$ .

**Lemma 1.4.6.** *Let  $\mathfrak{g}$  be a Lie algebra over an arbitrary field  $k$  and  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  a symmetric and associative bilinear form. For any ideal  $I \trianglelefteq \mathfrak{g}$  the orthogonal complement*

$$I^\perp := \{y \in \mathfrak{g} \mid \beta(x, y) = 0 \text{ for every } x \in I\}$$

*is also an ideal in  $\mathfrak{g}$ . In particular  $\text{rad } \beta = \mathfrak{g}^\perp$  is an Ideal in  $\mathfrak{g}$ .*

*Proof.* For  $z \in \mathfrak{g}$  and  $y \in I^\perp$  it follows for every  $x \in I$  that  $[x, y] = 0$  and thus

$$\beta(x, [y, z]) = \beta([x, y], z) = 0. \quad \square$$



**Remark 1.4.7.** The proof of Lemma 1.4.6 did not use that  $\beta$  is symmetric. This artificial restraint is only there to simplify the situation and notation (we do not need to distinguish between orthogonal complements from the left and from the right.) The main example of an associative bilinear form will be the Killing form, which is symmetric, so this assumption will pose no problems to us.

**Definition 1.4.8.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra over an arbitrary field  $k$ . The *Killing form* of  $\mathfrak{g}$  is the bilinear form

$$\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow k \quad \text{with} \quad \kappa(x, y) = \text{tr}(\text{ad}(x)\text{ad}(y)) \quad \text{for all } x, y \in \mathfrak{g}.$$

**Lemma 1.4.9.** *The Killing form  $\kappa$  of an finite dimensional Lie algebra  $\mathfrak{g}$  over an arbitrary field  $k$  is associative and symmetric.*

*Proof.* Recall from linear algebra that for any finite dimensional  $k$ -vector space  $V$  and all endomorphisms  $f_1, \dots, f_n \in \text{End}_k(V)$

$$\text{tr}(f_1 \cdots f_n) = \text{tr}(f_2 \cdots f_n f_1).$$

For all  $x, y \in \mathfrak{g}$  it follows that

$$\kappa(x, y) = \text{tr}(\text{ad}(x)\text{ad}(y)) = \text{tr}(\text{ad}(y)\text{ad}(x)) = \kappa(y, x),$$

so  $\kappa$  is symmetric. For all  $x, y, z$  it follows that

$$\begin{aligned} \kappa(x, [y, z]) &= \text{tr}(\text{ad}(x)\text{ad}([y, z])) = \text{tr}(\text{ad}(x)[\text{ad}(y), \text{ad}(z)]) \\ &= \text{tr}(\text{ad}(x)(\text{ad}(y)\text{ad}(z) - \text{ad}(z)\text{ad}(y))) \\ &= \text{tr}(\text{ad}(x)\text{ad}(y)\text{ad}(z)) - \text{tr}(\text{ad}(x)\text{ad}(z)\text{ad}(y)) \\ &= \text{tr}(\text{ad}(x)\text{ad}(y)\text{ad}(z)) - \text{tr}(\text{ad}(y)\text{ad}(x)\text{ad}(z)) \\ &= \text{tr}((\text{ad}(x)\text{ad}(y) - \text{ad}(y)\text{ad}(x))\text{ad}(z)) \\ &= \text{tr}([\text{ad}(x), \text{ad}(y)]\text{ad}(z)) = \text{tr}(\text{ad}([x, y])\text{ad}(z)) = \kappa([x, y], z). \quad \square \end{aligned}$$

**Example 1.4.10.** Let  $\mathfrak{g} := \mathfrak{gl}_n(k)$  for some arbitrary field  $k$ . Then

$$\kappa(x, y) = 2n \text{tr}(xy) - 2(\text{tr } x)(\text{tr } y) =: \beta(x, y) \quad \text{for all } x, y \in \mathfrak{g}.$$

To see this let  $(e_{ij})_{i,j=1,\dots,n}$  be the standard basis of  $\mathfrak{gl}_n(k)$ , i.e. the  $(i, j)$ -th entry of  $e_{ij}$  is 1, all other entries are 0 (hence  $e_{ij}e_k = \delta_{jk}e_i$  for every  $j = 1, \dots, n$  where  $(e_1, \dots, e_n)$  is the standard basis of  $k^n$ ). In particular

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \text{for all } i, j, k, l = 1, \dots, n. \quad (3)$$

To show that  $\kappa = \beta$  is sufficies to show that

$$\kappa(e_{ij}, e_{kl}) = \beta(e_{ij}, e_{kl}) \quad \text{for all } i, j, k, l = 1, \dots, n,$$

as both  $\kappa$  and  $\beta$  are bilinear forms on  $\mathfrak{g}$ . For all  $k, l, g, h = 1, \dots, n$  it follows from (3) that

$$\text{ad}(e_{kl})(e_{gh}) = [e_{kl}, e_{gh}] = e_{kl}e_{gh} - e_{gh}e_{kl} = \delta_{lg}e_{kh} - \delta_{kh}e_{gl}.$$

It follows that for all  $i, j, k, l, g, h = 1, \dots, n$

$$\begin{aligned} \text{ad}(e_{ij}) \text{ad}(e_{kl})(e_{gh}) &= \text{ad}(e_{ij})(\delta_{lg}e_{kh} - \delta_{kh}e_{gl}) = \delta_{lg} \text{ad}(e_{ij})(e_{kh}) - \delta_{kh} \text{ad}(e_{ij})(e_{gl}) \\ &= \delta_{lg}(\delta_{jk}e_{ih} - \delta_{ih}e_{kj}) - \delta_{kh}(\delta_{jg}e_{il} - \delta_{il}e_{gj}) \\ &= \delta_{jk}\delta_{lg}e_{ih} - \delta_{ih}\delta_{lg}e_{kj} - \delta_{jg}\delta_{kh}e_{il} + \delta_{il}\delta_{kh}e_{gj} \end{aligned}$$

and the coefficient of  $e_{gh}$  in this expression is

$$a_{gh} = \delta_{jk}\delta_{lg}\delta_{ig} + \delta_{il}\delta_{kh}\delta_{jh} - \delta_{ih}\delta_{lg}\delta_{kg}\delta_{jh} - \delta_{jg}\delta_{kh}\delta_{ig}\delta_{hl}.$$

It follows that for all  $i, j, k, l = 1, \dots, n$

$$\begin{aligned} \kappa(e_{ij}, e_{kl}) &= \sum_{g,h=1}^n (\delta_{jk}\delta_{lg}\delta_{ig} + \delta_{il}\delta_{kh}\delta_{jh} - \delta_{ih}\delta_{lg}\delta_{kg}\delta_{jh} - \delta_{jg}\delta_{kh}\delta_{ig}\delta_{hl}) \\ &= \sum_{g,h=1}^n \delta_{jk}\delta_{lg}\delta_{ig} + \sum_{g,h=1}^n \delta_{il}\delta_{kh}\delta_{jh} - \sum_{g,h=1}^n \delta_{ih}\delta_{lg}\delta_{kg}\delta_{jh} - \sum_{g,h=1}^n \delta_{jg}\delta_{kh}\delta_{ig}\delta_{hl} \\ &= n\delta_{jk}\delta_{il} + n\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl} = 2n\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} \\ &= 2n\delta_{jk}(\text{tr } e_{il}) - 2(\text{tr } e_{ij})(\text{tr } e_{kl}) = 2n \text{tr}(e_{ij}e_{kl}) - 2(\text{tr } e_{ij})(\text{tr } e_{kl}) \\ &= \beta(e_{ij}, e_{kl}). \end{aligned}$$

**Remark 1.4.11.** If  $\mathfrak{g}$  is a Lie algebra and  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  a finite dimensional representation of  $\mathfrak{g}$  then the corresponding trace form  $\phi_\rho$  is defined as

$$\phi_\rho(x, y) := \text{tr}(\rho(x)\rho(y)) \quad \text{for all } x, y \in \mathfrak{g}.$$

Replacing  $\text{ad}$  with  $\rho$  in the proof of Lemma 1.4.9 shows that  $\phi_\rho$  is an associative and symmetric bilinear form on  $\mathfrak{g}$ . The Killing form  $\kappa$  of  $\mathfrak{g}$  is then just the special case  $\kappa = \phi_{\text{ad}}$ .

**Lemma 1.4.12.** Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra over an arbitrary field  $k$ . Then for any ideal  $I \subseteq \mathfrak{g}$  the Killing form  $\kappa_I$  is given by restriction of the Killing form  $\kappa_{\mathfrak{g}}$  to  $I$ , i.e.  $\kappa_I = \kappa_{\mathfrak{g}}|_{I \times I}$ .

*Proof.* Let  $x, y \in I$ . Then  $I$  is  $\text{ad}_{\mathfrak{g}}(x)$ -invariant. Let  $(x_1, \dots, x_r)$  be a basis of  $I$  and  $(x_1, \dots, x_r, x_{r+1}, \dots, x_s)$  one of  $\mathfrak{g}$ . With respect to the basis  $(x_1, \dots, x_r)$  of  $I$  the endomorphism  $\text{ad}_I(x)$  is represented by a matrix  $A_x \in M_r(k)$  and  $\text{ad}_I(y)$  is represented by a matrix  $A_y \in M_r(k)$ . As  $I \triangleleft \mathfrak{g}$  is an ideal it follows that  $\text{im } \text{ad}_{\mathfrak{g}}(x) \subseteq \mathfrak{g}$  and  $\text{im } \text{ad}_{\mathfrak{g}}(y) \subseteq \mathfrak{g}$ , which is why with respect to the basis  $(x_1, \dots, x_s)$  the endomorphism  $\text{ad}_{\mathfrak{g}}(x)$  and  $\text{ad}_{\mathfrak{g}}(y)$  are represented by matrices

$$C_x = \begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix} \in M_s(k) \quad \text{and} \quad C_y = \begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix} \in M_s(k)$$

for some matrices  $B_x, B_y \in M_{s-r, r}(k)$ . Hence

$$\begin{aligned} \kappa_{\mathfrak{g}}(x, y) &= \text{tr}(\text{ad}_{\mathfrak{g}}(x) \text{ad}_{\mathfrak{g}}(y)) = \text{tr} \left( \begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} A_x A_y & A_x B_y \\ 0 & 0 \end{pmatrix} = \text{tr}(A_x A_y) = \text{tr}(\text{ad}_I(x) \text{ad}_I(y)) = \kappa_I(x, y). \quad \square \end{aligned}$$

**Example 1.4.13.** Let  $\mathfrak{g} := \mathfrak{sl}_n(k) = [\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)]$ . As seen in example 1.4.10 the Killing form of  $\mathfrak{gl}_n(k)$  is given by

$$\kappa_{\mathfrak{gl}_n(k)}(x, y) = 2n \operatorname{tr}(xy) - 2(\operatorname{tr} x)(\operatorname{tr} y) \quad \text{for all } x, y \in \mathfrak{gl}_n(k).$$

Because  $\mathfrak{sl}_n(k) \subseteq \mathfrak{gl}_n(k)$  it follows from Lemma 1.4.12 that the Killing form of  $\mathfrak{sl}_n(k)$  is given by

$$\kappa_{\mathfrak{sl}_n(k)}(x, y) = 2n \operatorname{tr}(xy) \quad \text{for all } x, y \in \mathfrak{sl}_n(k).$$

In particular the Killing form of  $\mathfrak{sl}_n(k)$  is just a multiple of the trace form.

**Lemma 1.4.14.** Let  $\mathfrak{g}$  be a Lie algebra and  $I_1, I_2 \subseteq \mathfrak{g}$  ideals with  $\mathfrak{g} = I_1 \oplus I_2$ . Then  $I_1 \perp I_2$  with respect to the Killing form  $\kappa$  of  $\mathfrak{g}$ . In particular it follows that for all  $x, y \in \mathfrak{g}$  with  $x = x_1 + x_2$  and  $y = y_1 + y_2$  with respect to  $\mathfrak{g} = I_1 \oplus I_2$

$$\kappa(x, y) = \kappa_{I_1}(x_1, y_1) + \kappa_{I_2}(x_2, y_2)$$

*Proof.* Because  $[I_1, I_2] \subseteq I_1 \cap I_2 = 0$  it follows that for every  $z_1 \in I_1$  and  $z_2 \in I_2$

$$(\operatorname{ad}(z_1)\operatorname{ad}(z_2))(\mathfrak{g}) = \operatorname{ad}(z_1)(\operatorname{ad}(z_2)(\mathfrak{g})) \subseteq \operatorname{ad}(z_1)(I_2) = 0.$$

Therefore  $\operatorname{ad}(z_1)\operatorname{ad}(z_2) = 0$  and in particular  $\kappa(z_1, z_2) = \operatorname{tr}(\operatorname{ad}(z_1)\operatorname{ad}(z_2)) = 0$ . From Lemma 1.4.12 it further follows that

$$\begin{aligned} \kappa(x, y) &= \kappa(x_1, y_1) + \kappa(x_1, y_2) + \kappa(x_2, y_1) + \kappa(x_2, y_2) \\ &= \kappa(x_1, y_1) + \kappa(x_2, y_2) = \kappa_{I_1}(x_1, y_1) + \kappa_{I_2}(x_2, y_2). \end{aligned} \quad \square$$

## 1.4.2. The concrete Jordan decomposition

**Definition 1.4.15.** Let  $V$  be an  $n$ -dimensional  $k$ -vector space and  $x \in \operatorname{End}_k(V)$  (resp.  $y \in M_n(y)$ ). Then  $x$  (resp.  $y$ ) is called *semisimple* if it is diagonalizable.

**Remark 1.4.16.** An endomorphism  $x \in \operatorname{End}_k(V)$  as above is semisimple if and only if every  $x$ -invariant subspace of  $V$  has a direct summand which is also  $x$ -invariant. (This depends on  $k$  being algebraically closed.)

**Theorem 1.4.17.** Let  $V$  be a finite dimensional  $k$ -vector space and  $x \in \operatorname{End}_k(V)$ .

1. There exist unique  $x_s, x_n \in \operatorname{End}_k(V)$  satisfying the following properties:
  - a)  $x = x_s + x_n$ .
  - b)  $x_s$  is semisimple and  $x_n$  is nilpotent.
  - c)  $x_s$  and  $x_n$  commute.
2.  $x_s$  and  $x_n$  are Polynomials in  $x$  without constant term, i.e. there exist polynomials  $P, Q \in k[T]$  such that  $P(0) = Q(0) = 0$  and  $x_s = P(x)$  and  $x_n = Q(x)$ . In particular an endomorphism of  $V$  commutes with  $x$  if and only if it commutes with  $x_s$  and  $x_n$ .

3. If  $A \subseteq B \subseteq V$  are linear subspaces with  $x(B) \subseteq A$  then also  $x_s(B) \subseteq A$  and  $x_n(B) \subseteq A$ .

*Proof.* Let  $\chi(T)$  be the characteristic polynomial of  $x$  with  $\chi(T) = \prod_{i=1}^n (T - \lambda_i)^{m_i}$  where  $\lambda_i \neq \lambda_j$  for  $i \neq j$ . By the chinese remainder theorem the map

$$\begin{aligned} k[T]/(\chi) &\longrightarrow \prod_{i=1}^n k[T]/((T - \lambda_i)^{m_i}), \\ F + (\chi) &\longmapsto (F + ((T - \lambda_1)^{m_1}), \dots, F + ((T - \lambda_n)^{m_n})) \end{aligned} \quad (4)$$

is surjective. Thus there exists some polynomial  $P \in k[T]$  with

$$P(T) \equiv \lambda_i \pmod{(T - \lambda_i)^{m_i}} \quad \text{for ever } i = 1, \dots, n. \quad (5)$$

We can also assume that  $P(0) = 0$ . If  $\lambda_i = 0$  for some  $i$  then this follows directly from (5). Otherwise the polynomials  $(T - \lambda_1)^{m_1}, \dots, (T - \lambda_n)^{m_n}, T$  are pairwise coprime, so by replacing  $\chi(T)$  with  $\tilde{\chi}(T) := \chi(T)T$  in (4) results in a polynomial  $\tilde{P}$  which does not only satisfy (5) (with  $P$  replaced by  $\tilde{P}$ ) but also  $\tilde{P} \bmod T = 0$ .

Now let  $Q(T) := T - P(T)$  as well as  $x_s := P(x)$  and  $x_n := Q(x)$ . Then  $x = x_s + x_n$  and  $x_s$  and  $x_n$  commute, as both are polynomials in  $x$ . For every  $i = 1, \dots, n$  let

$$V_i := \ker(x - \lambda_i)^{m_i}$$

be the generalized eigenspace of  $x$  with respect to the eigenvalue  $\lambda_i$ . Is is known from linear algebra that  $V = \bigoplus_{i=1}^n V_i$ .

It follows from (4) that for every  $i = 1, \dots, n$  there exists some polynomial  $P_i \in k[T]$  with

$$P(T) = \lambda_i + P_i(T)(T - \lambda_i)^{m_i},$$

from which follows for every  $v \in V_i$  that

$$x_s(v) = (\lambda_i \text{id}_V + P_i(x)(x - \lambda_i)^{m_i})(v) = \lambda_i v + \underbrace{P_i(x)((x - \lambda_i)^{m_i}(v))}_{=0} = \lambda_i v.$$

Hence  $V_i$  is  $x_s$ -invariant with  $x_s|_{V_i} = \lambda_i \text{id}_{V_i}$  for every  $i = 1, \dots, n$ . As  $V = \bigoplus_{i=1}^n V_i$  this shows that  $x_s$  is semisimple and  $V_i$  is precisely the eigenspace of  $x_s$  to the eigenvalue  $\lambda_i$ .

To see that  $x_s$  is nilpotent notice that for every  $i = 1, \dots, n$  and  $v \in V_i$

$$x_n(v) = x(v) - x_s(v) = x(v) - \lambda_i(v) = (x - \lambda_i \text{id}_V)(v).$$

Hence  $V_i$  is  $x_n$ -invariant with  $x_n|_{V_i} = x - \lambda_i \text{id}_{V_i}$  for every  $i = 1, \dots, n$ . By definition of  $V_i$  it follows that  $x_n|_{V_i}$  is nilpotent for every  $i = 1, \dots, n$ . Because  $V = \bigoplus_{i=1}^n V_i$  it follows that  $x_n$  is nilpotent.

This shows the existence of the claimed decomposition. For the uniqueness let  $y_s, y_n \in \text{End}_k(V)$  be any two endomorphisms with  $x = y_s + y_n$  where  $y_s$  is semisimple,  $y_n$  is nilpotent and  $y_s$  and  $y_n$  commute. As  $y_s$  and  $y_n$  commute it follows that each of

them commutes with  $x = y_s + y_n$ . Because  $x_s$  and  $x_n$  are polynomials in  $x$  it follows from this that  $y_s$  and  $y_n$  both commute with  $x_s$  and  $x_n$ . Hence  $x_s, x_n, y_s$  and  $y_n$  are pairwise commuting. In particular  $x_s$  and  $y_s$  are simultaneously diagonalizable which is why  $x_s - y_s$  is also semisimple. As  $x_n$  and  $y_n$  commute and are both nilpotent it also follows that  $y_n - x_n$  is nilpotent. But from  $x_s + x_n = x = y_s + y_n$  it follows that

$$\underbrace{x_s - y_s}_{\text{semisimple}} = \underbrace{y_n - x_n}_{\text{nilpotent}}.$$

Hence  $x_s - y_s = y_n - x_n = 0$ .

All other statements of the theorem directly follow from the construction of  $x_s$  and  $x_n$  and the fact that they are polynomials without constant term in  $x$ .  $\square$

**Remark 1.4.18.** An analogous statement of Theorem 1.4.17 can be shown for  $M_n(k)$  instead of  $\text{End}_k(V)$ . More precisely: Every matrix  $x \in M_n(k)$  can be uniquely decomposed into  $x = x_s + x_n$  such that  $x_s$  is semisimple,  $x_n$  is nilpotent and  $x_s$  and  $x_n$  commute. Both  $x_s$  and  $x_n$  are polynomials without constant term in  $x$ , so any matrix in  $M_n(k)$  commutes with  $x$  if and only if it commutes with  $x_s$  and  $x_n$ . If  $A \subseteq B \subseteq k^n$  are linear subspaces such that  $B$  is carried into  $A$  by left multiplication with  $x$ , then the same goes for  $x_s$  and  $x_n$ .

**Definition 1.4.19.** Let  $V$  be an  $n$ -dimensional vector space and  $x \in \text{End}_k(V)$  (resp.  $y \in M_n(k)$ ). Then the decomposition  $x = x_s + x_n$  from Theorem 1.4.17 (resp. the decomposition  $y = y_s + y_n$  from Remark 1.4.18) is called the *concrete Jordan decomposition* of  $x$  (resp.  $y$ ). The element  $x_s$  (resp.  $y_s$ ) is called the *semisimple part* of  $x$  (resp.  $y$ ) and the element  $x_n$  (resp.  $y_n$ ) is called the *nilpotent part* of  $x$  (resp.  $y$ ).

**Definition 1.4.20.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. Then  $x \in \mathfrak{g}$  is called *ad-semisimple* if  $\text{ad}(x)$  is a semisimple endomorphism of  $\mathfrak{g}$ .

**Lemma 1.4.21.** Let  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  be a Lie subalgebra for some finite dimensional vector space  $V$  (resp.  $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$ ) and  $x \in \mathfrak{g}$ .

1. If  $x$  is semisimple then  $x$  is also ad-semisimple.
2. If  $x$  is nilpotent then  $x$  is also ad-nilpotent.

*Proof.* We only show the case  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ , the proof for the case  $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$  being essentially the same.

1. Let  $e_1, \dots, e_n$  be a basis of  $\mathfrak{g}$  consisting of eigenvectors of  $x$ , where  $e_i$  belongs to the eigenvalue  $\lambda_i \in k$ . Then for all  $i, j = 1, \dots, n$  let  $E_{ij} \in \text{End}_k(\mathfrak{g})$  be defined by

$$E_{ij}(e_k) = \delta_{jk}e_i \quad \text{for every } k = 1, \dots, n.$$

Then  $(E_{ij})_{i,j=1,\dots,n}$  is a basis of  $\text{End}_k(\mathfrak{g})$ . For all  $i, j, k = 1, \dots, n$  it follows that

$$\begin{aligned} [x, E_{ij}](e_k) &= (xE_{ij} - E_{ij}x)(e_k) = x(E_{ij}(e_k)) - E_{ij}(x(e_k)) \\ &= \delta_{jk}x(e_i) - \lambda_k E_{ij}(e_k) = \delta_{jk}\lambda_i e_i - \delta_{jk}\lambda_k e_i \\ &= (\lambda_i - \lambda_j)\delta_{jk}e_i = (\lambda_i - \lambda_j)E_{ij}e_k. \end{aligned}$$

It follows that

$$\mathrm{ad}_{\mathfrak{gl}(V)}(x)(E_{ij}) = [x, E_{ij}] = (\lambda_i - \lambda_j)E_{ij} \quad \text{for all } i, j = 1, \dots, n,$$

so  $\mathrm{ad}_{\mathfrak{gl}(V)}$  is semisimple and therefore also the restriction  $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(x)|_{\mathfrak{g}}$ .

2. In Lemma 1.3.2 it was already shown that  $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$  is nilpotent. From this it follows that the restriction  $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(x)|_{\mathfrak{g}}$  is also nilpotent.  $\square$

**Corollary 1.4.22.** *Let  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  be a Lie subalgebra for a finite dimensional vector space  $V$  (resp.  $\mathfrak{g} = \mathfrak{gl}_n(k)$ ). If  $x \in \mathfrak{g}$  has the Jordan decomposition  $x = x_s + x_n$  then  $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$  is the Jordan decomposition of  $\mathrm{ad}(x)$ .*

*Proof.* As  $x_s$  is semisimple the same goes for  $\mathrm{ad}(x_s)$  and as  $x_n$  is nilpotent the same goes for  $\mathrm{ad}(x_n)$ , each following from Lemma 1.4.21. As  $x_s$  and  $x_n$  commute so do  $\mathrm{ad}(x_s)$  and  $\mathrm{ad}(x_n)$  because  $\mathrm{ad}$  is a homomorphism of Lie algebras.  $\square$

### 1.4.3. Cartan's Criterion

**Lemma 1.4.23.** *Let  $V$  be a finite dimensional  $k$ -vector space. Let  $A \subseteq B \subseteq \mathfrak{gl}(V)$  be linear subspaces and let*

$$T := \{z \in \mathfrak{gl}(V) \mid \mathrm{ad}(z)(B) \subseteq A\}.$$

*If  $x \in T$  and  $\mathrm{tr}(xz) = 0$  for every  $z \in T$  then  $x$  is nilpotent.*

*Proof.* Let  $x = x_s + x_n$  be the concrete Jordan decomposition of  $x$ . Then the concrete Jordan decomposition of  $\mathrm{ad}(x)$  is given by  $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$  by Corollary 1.4.22. As  $\mathrm{ad}(x)(B) \subseteq A$  it follows from the properties of the concrete Jordan decomposition (see Theorem 1.4.17) that also  $\mathrm{ad}(x_s)(B) = \mathrm{ad}(x)_s(B) \subseteq A$ . Hence  $x_s \in T$ .

Let  $(v_1, \dots, v_n)$  be an ordered basis of  $V$  with respect to which  $x$  is in Jordan normal form. Then with respect to this basis  $x_s$  is diagonal and  $x_n$  is strictly upper triangular. Let  $\lambda_i \in k$  with  $x_s(v_i) = \lambda_i v_i$  for every  $i = 1, \dots, n$  and set

$$E := \mathrm{span}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_n) \subseteq k.$$

To show that  $x$  is nilpotent it suffices to show that  $x_s = 0$ , which is equivalent to  $\lambda_i = 0$  for every  $i = 1, \dots, n$ . As this is the same as  $E = 0$  it is enough to show that  $f = 0$  for every  $\mathbb{Q}$ -linear map  $f: E \rightarrow \mathbb{Q}$ . For the rest of the proof we fix such an  $f$ . Let  $z: V \rightarrow V$  be defined by

$$z(v_i) := f(\lambda_i)v_i \quad \text{for every } i = 1, \dots, n.$$

**Claim.**  $z \in T$ .

*Proof.* For all  $i, j = 1, \dots, n$  let  $e_{ij} \in \mathfrak{gl}(V)$  with

$$e_{ij}(v_k) = \delta_{jk}v_i \quad \text{for every } k = 1, \dots, n.$$

Then  $(e_{ij})_{i,j=1,\dots,n}$  is a  $k$ -basis of  $\mathfrak{gl}(V)$ . As already seen in the proof of Corollary 1.4.22 this is a basis of eigenvectors of  $\text{ad}(x_s)$  where  $e_{ij}$  is an eigenvector of  $\text{ad}(x_s)$  with respect to the eigenvalue  $\lambda_i - \lambda_j$ . Because  $(v_1, \dots, v_n)$  is also a basis of  $V$  consisting of eigenvectors of  $z$ , where  $v_i$  belongs to the eigenvalue  $f(\lambda_i)$ , it follows in the same way, that  $e_{ij}$  is an eigenvector of  $\text{ad}(z)$  with respect to the eigenvalue

$$\mu_{ij} := f(\lambda_i) - f(\lambda_j) = f(\lambda_i - \lambda_j)$$

for all  $i, j = 1, \dots, n$ . In particular it follows that if  $e_{ij}$  and  $e_{i'j'}$  have the same eigenvalue with respect to  $\text{ad}(x_s)$ , i.e. if  $\lambda_i - \lambda_j = \lambda_{i'} - \lambda_{j'}$  then the same goes holds with respect to  $\text{ad}(z)$ . Hence if  $y \in \mathfrak{gl}(V)$  is an eigenvector of  $\text{ad}(x_s)$  with respect to the eigenvalue  $\lambda$  then  $y$  is an eigenvector of  $\text{ad}(z)$  with respect to the eigenvalue  $f(y)$ .

As  $\text{ad}(x)(B) \subseteq A \subseteq B$  there exists a decomposition  $B = A \oplus N$  into linear subspaces with  $A$  being  $\text{ad}(x)$ -invariant and thus decomposing into  $\text{ad}(x)$ -eigenspaces and  $N \subseteq \ker \text{ad}(x)$ . Then by the previous observations it follows that  $A$  also decomposes into  $\text{ad}(z)$ -eigenspaces and that  $\text{ad}(z)(N) = 0$ . Hence  $\text{ad}(z)(B) \subseteq A$  and thus  $z \in T$ .  $\square$

Now  $\text{tr}(xz) = \text{tr}(x_s z) + \text{tr}(x_n z)$ . As  $x_s$  and  $z$  are both diagonal with respect to the basis  $(v_1, \dots, v_n)$  it follows that  $\text{tr}(x_s z) = \sum_{i=1}^n f(\lambda_i) \lambda_i$ , and as  $x_n z$  is also strictly upper triangular it also follows that  $\text{tr}(x_n z) = 0$ . Together with  $z \in T$  this results in

$$0 = \text{tr}(xz) = \sum_{i=1}^n f(\lambda_i) \lambda_i$$

Because  $f(\lambda_i) \in \mathbb{Q}$  and  $\lambda_i \in E$  for every  $i = 1, \dots, n$  applying  $f$  to this equation results in

$$0 = \sum_{i=1}^n f(\lambda_i)^2.$$

It follows that  $f(\lambda_i) = 0$  for every  $i = 1, \dots, n$  and thus  $f = 0$ .  $\square$

**Remark 1.4.24.** The proof of Lemma 1.4.23 was not actually given in the lecture itself and proving it was an exercise on the third exercise sheet. The proof given above is the one I came up with based on some hints given on the exercise sheet. At some point it will properly be merged with the proof given in [Hum72, §4.3].

In the lecture a proof was given for the special case  $k = \mathbb{C}$ . But since I have some trouble understanding the details it is not included here (yet).

**Lemma 1.4.25** (Cartan's criterion for  $\mathfrak{gl}(V)$ ). *Let  $V$  be a finite dimensional  $k$ -vector space and  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  a Lie subalgebra. Then  $\mathfrak{g}$  is solvable if and only if  $\text{tr}(xy) = 0$  for every  $x \in \mathfrak{g}$  and  $y \in [\mathfrak{g}, \mathfrak{g}]$ .*

*Proof.* Suppose that  $\mathfrak{g}$  is solvable. Then by Lie's theorem there exists a basis of  $V$  with respect to which  $\mathfrak{g}$  is represented by upper triangular matrices. Then  $[\mathfrak{g}, \mathfrak{g}]$  is represented by strictly upper triangular matrices, which is why  $xy$  is also represented by a strictly upper triangular matrix for every  $x \in \mathfrak{g}$  and  $y \in [\mathfrak{g}, \mathfrak{g}]$ . Hence  $\text{tr}(xy) = 0$  for every  $x \in \mathfrak{g}$  and  $y \in [\mathfrak{g}, \mathfrak{g}]$ .

Now suppose that  $\text{tr}(xy) = 0$  for every  $x \in \mathfrak{g}$  and  $y \in [\mathfrak{g}, \mathfrak{g}]$ . Set  $A := [\mathfrak{g}, \mathfrak{g}]$ ,  $B := \mathfrak{g}$  and

$$T := \{x \in \mathfrak{gl}(V) \mid \text{ad}(x)(B) \subseteq A\}.$$

Let  $x \in [\mathfrak{g}, \mathfrak{g}] \subseteq T$  and  $z \in T$ . Then  $[z, \mathfrak{g}] = [z, B] \subseteq A = [\mathfrak{g}, \mathfrak{g}]$ . Writing  $x$  as  $x = \sum_{i=1}^n [a_i, b_i]$  with  $a_i, b_i \in \mathfrak{g}$  for every  $i = 1, \dots, n$  it follows that

$$\begin{aligned} \text{tr}(xz) &= \sum_{i=1}^n \text{tr}([a_i, b_i]z) = \sum_{i=1}^n \kappa_{\mathfrak{gl}(V)}([a_i, b_i], z) \\ &= \sum_{i=1}^n \kappa_{\mathfrak{gl}(V)}(a_i, [b_i, z]) = \sum_{i=1}^n \text{tr}(a_i \underbrace{[b_i, z]}_{\in [\mathfrak{g}, \mathfrak{g}]}) = 0, \end{aligned}$$

where the last step uses the assumption. It follows from Lemma 1.4.23 that  $x$  is nilpotent. Because  $[\mathfrak{g}, \mathfrak{g}]$  consists of nilpotent elements there exists a basis of  $V$  with respect to which  $[\mathfrak{g}, \mathfrak{g}]$  is represented by strictly upper triangular matrices. Hence  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent and  $\mathfrak{g}$  therefore solvable.  $\square$

**Theorem 1.4.26** (Cartan's criterion for solvability). *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. Then  $\mathfrak{g}$  is solvable if and only if*

$$\kappa(x, y) = 0 \quad \text{for every } x \in \mathfrak{g} \text{ and } y \in [\mathfrak{g}, \mathfrak{g}].$$

*Proof.* Because  $Z(\mathfrak{g})$  is a solvable ideal in  $\mathfrak{g}$  it follows that  $\mathfrak{g}$  is solvable if and only if  $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{ad } \mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$  is solvable. By Cartan's criterion for  $\mathfrak{gl}(\mathfrak{g})$  this is the case if and only if

$$\text{tr}(xy) = 0 \quad \text{for every } x \in \text{ad } \mathfrak{g} \text{ and } y \in [\text{ad}(\mathfrak{g}), \text{ad}(\mathfrak{g})].$$

Because  $[\text{ad}(\mathfrak{g}), \text{ad}(\mathfrak{g})] = \text{ad}([\mathfrak{g}, \mathfrak{g}])$  and  $\text{tr}(\text{ad}(x) \text{ad}(y)) = \kappa(x, y)$  for all  $x, y \in \mathfrak{g}$  this is equivalent to

$$\kappa(x, y) = 0 \quad \text{for every } x \in \mathfrak{g} \text{ and } y \in [\mathfrak{g}, \mathfrak{g}]. \quad \square$$

**Corollary 1.4.27.** *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra and  $\kappa$  the Killing form of  $\mathfrak{g}$ . Then  $\text{rad } \kappa$  is a solvable ideal of  $\mathfrak{g}$ . In particular  $\text{rad } \kappa \subseteq \text{rad } \mathfrak{g}$ .*

*Proof.* Lemma 1.4.6 already showed that  $\text{rad } \kappa$  is an ideal in  $\mathfrak{g}$ . From Lemma 1.4.12 and the definition of  $\text{rad } \kappa$  it follows that

$$\kappa_{\text{rad } \mathfrak{g}}(x, y) = \kappa(x, y) = 0 \quad \text{for all } x, y \in \text{rad } \kappa.$$

Hence by Cartan's criterion  $\text{rad } \kappa$  is solvable.  $\square$

## 1.5. The universal enveloping algebra

The following hold for this section alone: We fix an arbitrary field  $k$  and a Lie algebra  $\mathfrak{g}$  over  $k$ . By a  $k$ -algebra we always mean an associative and unitary one, and homomorphisms of  $k$ -algebras have to respect the unit. The resulting category of  $k$ -algebras with homomorphisms of  $k$ -algebras between them will be denoted by  $k\text{-Alg}$ .



### 1.5.1. Definition, properties and construction

**Definition 1.5.1.** An *universal enveloping algebra* of  $\mathfrak{g}$  is a  $k$ -algebra  $\mathcal{U}(\mathfrak{g})$  together with a homomorphism of Lie algebras  $\iota: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$  such that for every  $k$ -Algebra  $A$  and homomorphism of Lie algebras  $\phi: \mathfrak{g} \rightarrow A$  there exists a unique homomorphism of  $k$ -algebras  $\Phi: \mathcal{U}(\mathfrak{g}) \rightarrow A$  with  $\phi = \Phi \circ \iota$ , i.e. making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{g} & & \\ \downarrow \iota & \searrow \phi & \\ \mathcal{U}(\mathfrak{g}) & \xrightarrow{\Phi} & A \end{array}$$

**Remark 1.5.2.** As always with universal objects any two enveloping algebras of  $\mathcal{U}(\mathfrak{g})_1$  with  $\iota_1: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})_1$  and  $\mathcal{U}(\mathfrak{g})_2$  with  $\iota_2: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})_2$  of  $\mathfrak{g}$  are isomorphic, and there exists a unique isomorphism  $\varphi: \mathcal{U}(\mathfrak{g})_1 \rightarrow \mathcal{U}(\mathfrak{g})_2$  with  $\iota_2 = \varphi \circ \iota_1$ , i.e. making the following diagram commute:

$$\begin{array}{ccc} & \mathfrak{g} & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ \mathcal{U}(\mathfrak{g})_1 & \xrightarrow{\varphi} & \mathcal{U}(\mathfrak{g})_2 \end{array}$$

Hence we will talk about *the* universal enveloping algebra of  $\mathfrak{g}$ .

**Proposition 1.5.3.** Let  $V$  be a vector space over  $k$ . Then there exists a bijection

$$\begin{aligned} \left\{ \begin{array}{l} \text{Representations of } \mathfrak{g} \\ \rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V) \end{array} \right\} &\longrightarrow \left\{ \begin{array}{l} \mathcal{U}(\mathfrak{g})\text{-Modulstrukturen} \\ \theta: \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}_k(V) \end{array} \right\}, \\ \rho &\longmapsto \hat{\rho}, \\ \theta|_{\mathfrak{g}} &\longleftarrow \theta, \end{aligned}$$

where  $\hat{\rho}: \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}_k(V)$  is the  $k$ -algebra homomorphism induced by the homomorphism of Lie algebras  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  via the universal property of the universal enveloping algebra.

*Proof.* This is a direct consequence of the universal property of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ . □

**Remark 1.5.4.** By Proposition 1.5.3 the category of representations of  $\mathfrak{g}$  is isomorphic to the category of modules over the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ .

**Remark 1.5.5.** Given any two  $k$ -Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  then any homomorphism of Lie algebras  $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  induces a homomorphism of  $k$ -algebras  $\phi^*: \mathcal{U}(\mathfrak{g}_1) \rightarrow \mathcal{U}(\mathfrak{g}_2)$  via the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{g}_1 & \xrightarrow{\varphi} & \mathfrak{g}_2 \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
\mathcal{U}(\mathfrak{g}_1) & \xrightarrow{\varphi^*} & \mathcal{U}(\mathfrak{g}_2)
\end{array}$$

Hence the assignment  $\mathfrak{g} \mapsto \mathcal{U}(\mathfrak{g})$  of a Lie algebra to its universal enveloping algebra can be extended to a (covariant) functor  $\mathcal{U}: k\text{-Lie} \rightarrow k\text{-Alg}$ . It is by the universal property of the universal enveloping algebra left adjoint to the functor  $k\text{-Alg} \rightarrow k\text{-Lie}$  which assigns each  $k$ -algebra its Lie algebra.

**Lemma 1.5.6.** *Let  $T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$  be the tensor algebra and  $\mathcal{I} \subseteq T(\mathfrak{g})$  the two-sided ideal generated by the element  $x \otimes y - y \otimes x - [x, y]$  with  $x, y \in \mathfrak{g}$ . The quotient  $\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/\mathcal{I}$  together with the  $k$ -linear map*

$$\iota: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g}), \quad x \mapsto x + \mathcal{I}$$

*is an universal enveloping algebra of  $\mathfrak{g}$ .*

*Proof.*  $\mathcal{U}(\mathfrak{g})$  is a  $k$ -algebra by construction and  $\iota$  is a homomorphism of Lie algebras since for all  $x, y \in \mathfrak{g}$

$$\begin{aligned}
[\iota(x), \iota(y)] &= [x + \mathcal{I}, y + \mathcal{I}] = (x + \mathcal{I})(y + \mathcal{I}) - (y + \mathcal{I})(x + \mathcal{I}) \\
&= (x \otimes y - y \otimes x) + \mathcal{I} = [x, y] + \mathcal{I} = \iota([x, y]).
\end{aligned}$$

Given any  $k$ -algebra  $A$  and homomorphism of Lie algebras  $\phi: \mathfrak{g} \rightarrow A$  it can be uniquely extended to a homomorphism of  $k$ -algebras  $\hat{\phi}: T(\mathfrak{g}) \rightarrow A$  via

$$\hat{\phi}(x_1 \otimes \cdots \otimes x_n) = \phi(x_1) \cdots \phi(x_n) \quad \text{for all } n \geq 0 \text{ and } x_1, \dots, x_n \in \mathfrak{g}.$$

Because  $\phi$  is not only  $k$ -linear but even a homomorphism of Lie algebras it follows that for all  $x, y \in \mathfrak{g}$

$$\hat{\phi}(x \otimes y - y \otimes x) = \phi(x)\phi(y) - \phi(y)\phi(x) = [\phi(x), \phi(y)] = \phi([x, y]) = \hat{\phi}([x, y])$$

It follows that  $\hat{\phi}(x) = 0$  for every  $x \in \mathcal{I}$ . Hence  $\hat{\phi}$  factors through a unique homomorphism of  $k$ -algebras

$$\Phi: \mathcal{U}(\mathfrak{g}) \rightarrow A, \quad x_1 \otimes \cdots \otimes x_n + \mathcal{I} \mapsto \phi(x_1) \cdots \phi(x_n)$$

for all  $n \geq 0$  and  $x_1, \dots, x_n \in \mathfrak{g}$ . For every  $x \in \mathfrak{g}$  it follows that

$$(\Phi \circ \iota)(x) = \Phi(\iota(x)) = \Phi(x + \mathcal{I}) = \phi(x),$$

which is why  $\phi = \Phi \circ \iota$ . That  $\Phi$  is the unique homomorphism of  $k$ -algebras with this properties follows from the uniqueness of  $\hat{\phi}$ .  $\square$

**Corollary 1.5.7.** *The homomorphism  $\iota: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$  is injective. As a  $k$ -algebra  $\mathcal{U}(\mathfrak{g})$  is generated by  $\iota(\mathfrak{g})$ .*

**Remark 1.5.8.** We will always identify  $\mathfrak{g}$  with its image under  $\iota$ .

## 1.5.2. Poincaré-Birkhoff-Witt

### 1.5.2.1. Graded $k$ -algebras

**Definition 1.5.9.** A *grading*, also called *gradation*, of a  $k$ -algebra  $A$  is a direct sum decomposition  $A = \bigoplus_{i \in \mathbb{N}} A_i$  into linear subspaces such that

$$A_i A_j \subseteq A_{i+j} \quad \text{for all } i, j \in \mathbb{N}.$$

A *graded  $k$ -algebra* is a  $k$ -algebra  $A$  together with a grading  $A = \bigoplus_{n \in \mathbb{N}} A_n$ .

**Remark 1.5.10.** While a graded  $k$ -algebra is formally a pair  $(A, (A_n)_{n \in \mathbb{N}})$  consisting of a  $k$ -algebra  $A$  and a grading  $A = \bigoplus_{n \in \mathbb{N}} A_n$  we will often just call  $A$  a graded  $k$ -algebra without explicitly mentioning the grading. We also set  $A_n := 0$  for every  $n < 0$ .

**Remark 1.5.11.** Given any semigroup  $(S, \cdot)$  an  $S$ -grading of a  $k$ -algebra  $A$  is a decomposition  $A = \bigoplus_{s \in S} A_s$  into linear subspaces such that  $A_s A_t \subseteq A_{s \cdot t}$  for all  $s, t \in S$ . An  $S$ -graded  $k$ -algebra is a  $k$ -algebra  $A$  together with an  $S$ -grading  $A = \bigoplus_{s \in S} A_s$ . An graded  $k$ -algebra in the sense of Definition 1.5.9 is then the special case of an  $\mathbb{N}$ -graded  $k$ -algebra.

**Lemma 1.5.12.** *Let  $A$  be a graded  $k$ -algebra. Then  $1 \in A_0$  and  $A_0$  is a  $k$ -subalgebra.*

*Proof.* Let  $1 = \sum_{i \in \mathbb{N}} e_i$  with respect to  $A = \bigoplus_{n \in \mathbb{N}} A_n$ . Then for any  $j \in \mathbb{N}$  and  $a \in A_j$

$$A_j \ni a = a \cdot 1 = a \left( \sum_{i \in \mathbb{N}} e_i \right) = \sum_{i \in \mathbb{N}} \underbrace{a e_i}_{\in A_{i+j}},$$

and it follows from the directness of the decomposition  $A = \bigoplus_{n \in \mathbb{N}} A_n$  that  $a = a e_0$ . It follows that  $a e_0 = a$  for every  $a \in A$ , hence  $e_0$  is the unit of  $A$ .

That  $A_0$  is a linear subspace which is closed under the multiplication follows from the definition of a graded  $k$ -algebra. As it contains the unit of  $A$  it is a  $k$ -subalgebra.  $\square$

**Examples 1.5.13.** 1. Any  $k$ -algebra  $A$  becomes a graded  $k$ -algebra by setting  $A_0 := A$  and  $A_i := 0$  for every  $i > 1$ .

2. The polynomial ring  $A = k[x_1, \dots, x_n]$  is a graded  $k$ -algebra by setting

$$A_d := \text{span}_k \{x_1^{p_1} \cdots x_n^{p_n} \mid p_1 + \cdots + p_n = d\} \quad \text{for every } d \in \mathbb{N},$$

i.e.  $A_d$  consists of the homogeneous polynomials of degree  $d$ .

3. Let  $V$  be a  $k$ -vector space. Then the tensor algebra  $T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$ , the symmetric algebra  $S(V) = \bigoplus_{n \in \mathbb{N}} S^n(V)$  and the exterior algebra  $\Lambda(V) = \bigoplus_{n \in \mathbb{N}} \Lambda^n(V)$  carry the structure of a graded  $k$ -algebra via  $T(V)_n := V^{\otimes n}$ ,  $S(V)_n := S^n(V)$  and  $\Lambda(V)_n := \Lambda^n(V)$  for every  $n \in \mathbb{N}$ .

**Definition 1.5.14.** Let  $A$  and  $B$  be graded  $k$ -algebras. A homomorphism of  $k$ -algebras  $\varphi: A \rightarrow B$  is called a *homomorphism of graded  $k$ -algebras* if  $\varphi(A_n) \subseteq B_n$  for every  $n \in \mathbb{N}$ , and the induced homomorphisms of vector spaces are denoted by  $\varphi_n: A_n \rightarrow B_n$  for every  $n \in \mathbb{N}$ . An homomorphism of graded  $k$ -algebras is called an isomorphism if it is bijective.

**Remark 1.5.15.** If  $A$  is a graded  $k$ -algebra then  $\text{id}_A$  is a homomorphism of graded  $k$ -algebras, and if  $B$  and  $C$  are two other graded  $k$ -algebras and  $\varphi: A \rightarrow B$  and  $\psi: B \rightarrow C$  homomorphisms of graded  $k$ -algebras then so is  $\psi \circ \varphi: A \rightarrow C$ . Hence the graded  $k$ -algebras together with the homomorphisms of graded  $k$ -algebras between them form a category, which will be referred to by  $k\text{-Grad}$ .

**Example 1.5.16.** 1. For any vector space  $V$  the two maps

$$T(V) \rightarrow S(V), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$$

and

$$T(V) \rightarrow \Lambda(V), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \wedge \cdots \wedge x_n$$

are homomorphisms of graded  $k$ -algebras.

2. If  $V$  is a finite dimensional vector space with basis  $x_1, \dots, x_n$  the the isomorphism of  $k$ -algebras

$$k[T_1, \dots, T_n] \rightarrow S(V), \quad T_i \mapsto x_i \quad \text{for every } i = 1, \dots, n$$

is already an isomorphism of graded  $k$ -algebras.

**Definition 1.5.17.** Let  $A$  be a graded  $k$ -algebra. A two-sided ideal  $J \subseteq A$  is called *homogeneous* if  $J = \bigoplus_{n \in \mathbb{N}} (J \cap A_n)$ . Equivalently, given any  $x \in J$  with the decomposition  $x = \sum_{n \in \mathbb{N}} x_n$  with respect to  $A = \bigoplus_{n \in \mathbb{N}} A_n$  it follows that  $x_n \in J$  for every  $n \in \mathbb{N}$ .

**Lemma 1.5.18.** Let  $A$  be a graded  $k$ -algebra and  $J \subseteq A$  a two-sided homogeneous ideal. Then  $A/J$  is a graded  $k$ -algebra via  $(A/J)_n = A_n / (J \cap A_n)$  for every  $n \in \mathbb{N}$  and the canonical projection  $\pi: A \rightarrow A/J, a \mapsto a + J$  is a homomorphism of graded  $k$ -algebras.

*Proof.* This follows directly from the definition of a homogeneous ideal.  $\square$

**Lemma 1.5.19.** Let  $A$  be a graded  $k$ -algebra,  $J \subseteq A$  a two-sided ideal and call an element  $x \in J$  *homogeneous (in  $J$ )* if  $x_n \in J$  for every  $n \in \mathbb{N}$  where  $x = \sum_{n \in \mathbb{N}} x_n$  with respect to  $A = \bigoplus_{n \in \mathbb{N}} A_n$ . Then  $J$  is homogeneous if and only if  $J$  is generated by elements which are homogeneous in  $J$ .

*Proof.* Let  $I := \{x \in J \mid x \text{ is homogeneous in } J\}$  be the linear subspace of elements which are homogeneous in  $J$ . If  $x \in I$  then  $x_n \in J$  for every  $n \in \mathbb{N}$  where  $x = \sum_{n \in \mathbb{N}} x_n$  with respect to  $A = \bigoplus_{n \in \mathbb{N}} A_n$ . Given  $a \in A$  with  $a = \sum_{m \in \mathbb{N}} a_m$  with respect to  $A = \bigoplus_{m \in \mathbb{N}} A_m$  it follows that  $a_m x_n \in J$  for all  $m, n \in \mathbb{N}$  because  $J$  is left ideal, and

therefore  $ax = \sum_{m,n \in \mathbb{N}} a_m x_n \in J$ . Hence  $I$  is a left ideal. In the same way it follows that  $I$  is also a right ideal and hence already a two-sided ideal in  $A$ .

The ideal  $J$  is homogeneous if and only if any of its elements is homogeneous in  $J$ , i.e. if  $I = J$ , from which the statement follows.  $\square$

**Examples 1.5.20.** 1. If  $A$  and  $B$  are graded  $k$ -algebras and  $\varphi: A \rightarrow B$  a homomorphism of graded  $k$ -algebras then  $\ker \varphi$  is a homogeneous ideal.

2. Let  $V$  be any vector space. The two-sided ideal  $I$  of  $T(V)$  generated by the elements  $x \otimes y - y \otimes x$  with  $x, y \in V$  is a homogeneous ideal of  $T(V)$ . The same goes for the two-sided ideal  $J$  generated by the elements  $x \otimes x$  with  $x \in V$ . The resulting (graded) quotient algebras are  $S(V)$  and  $\Lambda(V)$ .

### 1.5.2.2. Filtered $k$ -algebras

**Definition 1.5.21.** A *filtration* of a  $k$ -algebra  $A$  is an increasing sequence

$$A_{(0)} \subseteq A_{(1)} \subseteq A_{(2)} \subseteq \cdots \subseteq A$$

such that  $A = \bigcup_{i \in \mathbb{N}} A_{(i)}$  and  $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$  for all  $i, j \in \mathbb{N}$ , as well as  $1 \in A_{(0)}$ . A *filtered  $k$ -algebra* is a  $k$ -algebra  $A$  together with a filtration  $A = \bigcup_{n \in \mathbb{N}} A_{(n)}$ .

**Remark 1.5.22.** As for graded  $k$ -algebras we will refer to a filtered  $k$ -algebra  $A$  without explicitly mentioning the filtration. We also set  $A_{(n)} := 0$  for every  $n < 0$ .

**Definition 1.5.23.** Let  $A$  and  $B$  be filtered  $k$ -algebras. A homomorphism of  $k$ -algebras  $\varphi: A \rightarrow B$  is called a *homomorphism of filtered  $k$ -algebras* if  $\varphi(A_{(n)}) \subseteq B_{(n)}$  for every  $n \in \mathbb{N}$ .

**Remark 1.5.24.** If  $A, B$  and  $C$  are filtered  $k$ -algebras then  $\text{id}_A$  is a homomorphism of filtered  $k$ -algebras and if  $\varphi: A \rightarrow B$  and  $\psi: B \rightarrow C$  are homomorphisms of filtered  $k$ -algebras then  $\psi \circ \varphi$  is also a homomorphism of filtered  $k$ -algebras. It follows that filtered  $k$ -algebras together with homomorphisms of filtered  $k$ -algebras between them form a category, which will be referred to by  **$k$ -Filt**.

**Examples 1.5.25.** 1. Any graded  $k$ -algebra  $A$  also carries the structure of an filtered  $k$ -algebra by setting  $A_{(n)} := \bigoplus_{i \leq n} A_i$  for every  $n \in \mathbb{N}$ . If  $\varphi: A \rightarrow B$  is a homomorphism of graded  $k$ -algebras then it is also a homomorphism of filtered  $k$ -algebras. Hence this construction results into a functor  $\text{flt}: k\text{-Grad} \rightarrow k\text{-Filt}$ .

2. If  $A$  is a filtered algebra and  $I \subseteq A$  any two-sided ideal then  $A/I$  is a filtered  $k$ -algebra via  $(A/I)_{(n)} := \pi(A_{(n)})$  for every  $n \in \mathbb{N}$ , where  $\pi: A \rightarrow A/I, a \mapsto a + I$  denotes the canonical projection.

3. If  $\mathfrak{g}$  is a  $k$ -Lie algebra then  $\mathcal{U}(\mathfrak{g})$  carries the structure of a filtered  $k$ -algebra induced by the filtration of  $T(\mathfrak{g})$ , which in turn is induced by the gradation of  $T(\mathfrak{g})$ . Explicitly

$$\mathcal{U}(\mathfrak{g})_{(n)} = \text{span}_k \{x_1 \cdots x_m \mid m \in \mathbb{N}, m \leq n, x_1, \dots, x_m \in \mathfrak{g}\} \quad \text{for every } n \in \mathbb{N}.$$

**Lemma 1.5.26.** Let  $A$  be a graded  $k$ -algebra and set  $B_n := A_n/A_{n-1}$  for every  $n \in \mathbb{N}$ . Then for all  $n, m \in \mathbb{N}$  the map

$$B_n \times B_m \rightarrow B_{n+m}, ([a], [b]) \mapsto [ab]$$

is well-defined and bilinear.

*Proof.* Let  $a, a' \in A_n$  and  $b, b' \in A_m$  with  $[a] = [a']$  and  $[b] = [b']$ . Then  $ab, a'b' \in A_{n+m}$  and because  $[a] = [a']$  and  $[b] = [b']$  it follows that  $a - a' \in A_{n-1}$  and  $b - b' \in A_{m-1}$ . Therefore

$$ab = (a' + (a - a'))(b' + (b - b')) = a'b' + \underbrace{(a - a')b'}_{\in A_{n+m-1}} + \underbrace{a'(b - b')}_{\in A_{n+m-1}} + \underbrace{(a - a')(b - b')}_{\in A_{n+m-2}}$$

and thus  $[ab] = [a'b']$ .  $\square$

**Definition 1.5.27.** Let  $A$  be a filtered  $k$ -algebra. Its *associated graded  $k$ -algebra* is the graded  $k$ -algebra consisting of the underlying vector space  $\text{gr}(A) := \bigoplus_{n \in \mathbb{N}} \text{gr}_n(A)$  with  $\text{gr}_i(A) := A_{(i)}/A_{(i-1)}$  for every  $i \in \mathbb{N}$  and the multiplication  $\text{gr}(A) \times \text{gr}(A) \rightarrow \text{gr}(A)$  induced by the well-defined bilinear maps

$$\text{gr}_n(A) \times \text{gr}_m(A) \rightarrow \text{gr}_{n+m}(A), \quad ([a], [b]) \mapsto [ab] \quad \text{for all } n, m \in \mathbb{N}.$$

together with the grading given by  $\text{gr}(A)_n := \text{gr}_n(A)$  for every  $n \in \mathbb{N}$ .

**Remark 1.5.28.** If  $A$  and  $B$  are filtered  $k$ -algebras and  $\varphi: A \rightarrow B$  is a homomorphism of filtered  $k$ -algebras then  $\varphi$  induces a  $k$ -linear map  $\varphi_n: \text{gr}(A)_n \rightarrow \text{gr}(B)_n, [a] \mapsto [\varphi(a)]$  for every  $n \in \mathbb{N}$ , which result in a homomorphism of graded  $k$ -algebras

$$\text{gr}(\varphi) := \bigoplus_{n \in \mathbb{N}} \varphi_n: \text{gr}(A) \rightarrow \text{gr}(B), \quad \sum_{n \in \mathbb{N}} a_n \mapsto \sum_{n \in \mathbb{N}} \varphi(a_n).$$

Hence  $\text{gr}$  can be seen as a functor  $\text{gr}: k\text{-Filt} \rightarrow k\text{-Grad}$ .

**Examples 1.5.29.** 1. If  $A$  is a graded  $k$ -algebra then  $\text{gr}(\text{flt}(A))$  is naturally isomorphic to  $A$  in the following way: The filtration on  $\text{flt}(A)$  is given by  $A_{(n)} = \bigoplus_{i \leq n} A_i$  for every  $n \in \mathbb{N}$ . Hence there is for every  $n \in \mathbb{N}$  an isomorphism of vector spaces

$$\varphi_n: A_n \rightarrow A_{(n)}/A_{(n-1)} = \text{gr}(\text{flt}(A))_n, \quad a \mapsto [a].$$

Combining these isomorphisms results in an isomorphism of graded  $k$ -algebras

$$\bigoplus_{n \in \mathbb{N}} \varphi_n: \bigoplus_{n \in \mathbb{N}} A_n \rightarrow \bigoplus_{n \in \mathbb{N}} \text{gr}(\text{flt}(A))_n, \quad \sum_{n \in \mathbb{N}} a_n \mapsto \sum_{n \in \mathbb{N}} [a_n].$$

We will therefore identify  $A$  with  $\text{gr} A$  in the above way.

2. Let  $\mathfrak{g}$  be a  $k$ -Lie algebra. The canonical projection

$$\pi: T(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}), \quad x_1 \otimes \cdots \otimes x_n \rightarrow x_1 \cdots x_n \quad \text{for every } n \in \mathbb{N} \text{ and all } x_1, \dots, x_n \in \mathfrak{g}$$

is a homomorphism of filtered  $k$ -algebras, where the filtration of  $T(\mathfrak{g})$  is induced by the gradation discussed in Examples 1.5.13. Hence it induces a homomorphism of graded  $k$ -algebras

$$\text{gr}(\pi): T(\mathfrak{g}) \rightarrow \text{gr}(\mathcal{U}(\mathfrak{g})),$$

where  $T(\mathfrak{g})$  is identified with  $\text{gr}(T(\mathfrak{g}))$  as above. This homomorphism maps an element  $x_1 \otimes \cdots \otimes x_n$  with  $x_1, \dots, x_n \in \mathfrak{g}$  to the residue class  $[x_1 \cdots x_n] \in \text{gr}(\mathcal{U}(\mathfrak{g}))_n$ .

### 1.5.2.3. The Poincaré-Birkhoff-Witt theorem

For this subsection we fix some  $k$ -Lie algebra  $\mathfrak{g}$  with basis  $(x_i)_{i \in I}$  where  $(I, \leq)$  is a totally ordered index set. Before stating and proving the *Poincaré-Birkhoff-Witt theorem* (PBW) we fix some notation which we will only use in this subsection.

**Definition 1.5.30.** Let  $\mathcal{I}_n := \{(i_1, \dots, i_n) \mid i_1, \dots, i_n \in I, i_1 \leq \dots \leq i_n\}$  for every  $n \in \mathbb{N}$  and set  $\mathcal{I} := \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$ . For every  $\alpha = (i_1, \dots, i_n) \in \mathcal{I}$  with  $n \in \mathbb{N}$  let  $x_\alpha := x_{i_1} \cdots x_{i_n} \in \mathcal{U}(\mathfrak{g})$ .

**Remark 1.5.31.** Notice that  $\mathcal{I}_0$  contains the empty tuple.

**Theorem 1.5.32** (PBW (concrete version)). *The family  $(x_\alpha \mid \alpha \in \mathcal{I})$  is a  $k$ -basis of  $\mathcal{U}(\mathfrak{g})$ .*

**Remark 1.5.33.** The basis  $(x_\alpha \mid \alpha \in \mathcal{I})$  can also be written as

$$(x_{i_1}^{p_1} \cdots x_{i_n}^{p_n} \mid n \in \mathbb{N}, i_1, \dots, i_n \in I, i_1 < \dots < i_n, p_1, \dots, p_n \geq 1).$$

**Example 1.5.34.** If  $\mathfrak{g}$  is a finite dimensional  $k$ -Lie algebra with basis  $x_1, \dots, x_n$  then  $\mathcal{U}(\mathfrak{g})$  has a basis given by  $(x_1^{p_1} \cdots x_n^{p_n} \mid p_1, \dots, p_n \in \mathbb{N})$ . In particular a basis of  $\mathcal{U}(\mathfrak{sl}_2(k))$  is given by  $(e^\ell h^m f^n \mid \ell, m, n \in \mathbb{N})$ .

**Lemma 1.5.35.** *The collection  $(x_\alpha \mid \alpha \in \mathcal{I})$  generates  $\mathcal{U}(\mathfrak{g})$  as a vector space.*

*Proof.* To show that  $(x_\alpha \mid \alpha \in \mathcal{I})$  generates  $\mathcal{U}(\mathfrak{g})$  as a vector space it suffices to show that  $\mathcal{B}_n := (x_\alpha \mid m \leq n, \alpha \in \mathcal{I}_m)$  generates  $\mathcal{U}(\mathfrak{g})_{(n)}$  as a vector space, which will be shown by induction over  $n \in \mathbb{N}$ : For  $n = 0$  this is clear as  $\mathcal{U}(\mathfrak{g}) = k$  is one-dimensional and thus spanned by  $x_{()}$ , the monomial corresponding to the empty tuple.

Suppose that the statement holds for some  $n \in \mathbb{N}$ . Then  $\mathcal{U}(\mathfrak{g})_{(n+1)}$  is generated as a vector space as a by the monomials  $(x_{(i_1, \dots, i_n)} \mid m \leq n+1, i_1, \dots, i_n \in I)$ . Therefore it suffices to express these monomials in terms of  $\mathcal{B}_{n+1}$ . By induction hypothesis it is enough to check this for the monomials  $(x_{(i_1, \dots, i_{n+1})} \mid i_1, \dots, i_{n+1} \in I)$ . For this let  $\alpha = (i_1, \dots, i_{n+1})$  be some fixed multiindex with  $i_1, \dots, i_{n+1} \in I$ .

For any two  $x, y \in \mathfrak{g}$  one has  $xy = yx + [x, y]$  with  $[x, y] \in \mathcal{U}(\mathfrak{g})_{(1)}$ . Hence there exists for any permutation  $\sigma \in S_{n+1}$  a linear combination  $R_\sigma \in \mathcal{U}(\mathfrak{g})_{(1)}$  with

$$x_\alpha = x_{i_1} \cdots x_{i_{n+1}} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n+1)}} + R_\sigma = x_{(i_{\sigma(1)}, \dots, i_{\sigma(n+1)})} + R_\sigma.$$

Let  $\sigma \in S_{n+1}$  be a permutation with  $i_{\sigma(1)} \leq \dots \leq i_{\sigma(n+1)}$ . By induction hypothesis  $R_\sigma \in \mathcal{U}(\mathfrak{g})_{(n)}$  can be expressed as a linear combination of the monomials  $\mathcal{B}_n$ . Hence  $x_\alpha = x_{(i_{\sigma(1)}, \dots, i_{\sigma(n+1)})} + R_\sigma$  can be expressed as a linear combination of the monomials  $\mathcal{B}_{n+1}$  because  $x_{(i_{\sigma(1)}, \dots, i_{\sigma(n+1)})}$  is one of them.  $\square$

*Proof of PBW (concrete version).* By Lemma 1.5.35 the collection  $(x_\alpha \mid \alpha \in \mathcal{I})$  generates  $\mathcal{U}(\mathfrak{g})$  as a vector space, so all that's left to show is that it is linearly independent.

Let  $V := k[Z_i \mid i \in I]$  and for every  $n \in \mathbb{N}$  let  $V_{(n)}$  be the polynomials of degree  $\leq n$ . For every  $n \in \mathbb{N}$  and  $\alpha = (i_1, \dots, i_n) \in I^n$  write  $Z_\alpha := Z_{i_1} \cdots Z_{i_n}$ . If  $i \in I$  and  $\alpha = (i_1, \dots, i_n) \in I^n$  then write  $i \leq \alpha$  if  $i \leq i_j$  for every  $j = 1, \dots, n$ . Also set  $i \cdot \alpha = (i, i_1, \dots, i_n) \in I^{n+1}$ .

To show that  $(x_\alpha \mid \alpha \in \mathcal{I})$  is linearly independent  $V$  will be given the structure of a representation of  $\mathfrak{g}$  such that

$$x_i \cdot Z_\alpha = Z_{i \cdot \alpha} \quad \text{for every } i \in I \text{ and } \alpha \in \mathcal{I} \text{ with } i \leq \alpha.$$

Then for the corresponding  $\mathcal{U}(\mathfrak{g})$ -module structure on  $V$  it follows that

$$x_\alpha \cdot 1 = x_\alpha \cdot Z_{()} = Z_\alpha \quad \text{for every } \alpha \in \mathcal{I}$$

where  $1 \in V = k[Z_i \mid i \in I]$  and  $()$  denotes the empty tuple. Because  $(Z_\alpha \mid \alpha \in \mathcal{I})$  is linearly independent it then follows that  $(x_\alpha \mid \alpha \in \mathcal{I})$  is linearly independent. The existence of such an action follows from the following:

**Claim.** *There exists a unique sequence  $(\varphi_n)_{n \in \mathbb{N}}$  of bilinear maps*

$$\varphi_n : \mathfrak{g} \times V_{(n)} \rightarrow V_{(n+1)}, \quad (x, p) \mapsto x \cdot p$$

*satisfying the following conditions:*

1. *The restriction of  $\varphi_{n+1}$  to  $\mathfrak{g} \times V_{(n)}$  coincides with  $\varphi_n$  for every  $n \in \mathbb{N}$*
2.  *$x_i \cdot Z_\alpha = Z_{i \cdot \alpha}$  for every  $i \in I$  and  $\alpha \in \mathcal{I}$  with  $i \leq \alpha$ .*
3.  *$x_i \cdot x_j \cdot Z_\alpha - x_j \cdot x_i \cdot Z_\alpha = [x_i, x_j] \cdot Z_\alpha$  for all  $i, j \in I$  and every  $\alpha \in \mathcal{I}$ .*
4.  *$x_i \cdot Z_\alpha - Z_{i \cdot \alpha} \in V_{(n)}$  for every  $n \in \mathbb{N}$ ,  $i \in I$  and  $\alpha \in \mathcal{I}_n$ .*

*(Condition 1 actually follows from the other conditions by the uniqueness of the sequence  $(\varphi_n)_{n \in \mathbb{N}}$ . See [Hum72, §17.4] for more details.)*

*Proof.* Notice that the notation  $x \cdot p$  with  $x \in \mathfrak{g}$  and  $p \in V$  is unambiguous by condition 1. The maps  $\varphi_n$  will be defined by induction over  $n$ :

As  $V_{(0)}$  is one-dimensional and spanned by  $1 = Z_{()}$  it follows from condition 2 that  $x_i \cdot 1 = x_i \cdot Z_{()} = Z_{i \cdot ()} = Z_i$  for every  $i \in I$ . This defines  $\varphi_0$  uniquely. Conditions 2 and 4 hold by construction and the conditions 1 and 3 do not affect  $\varphi_0$ .

Let  $n \in \mathbb{N}$  and suppose  $\varphi_m$  is constructed for every  $m \leq n$ . By condition 1 all that is left to define is  $x_i \cdot Z_\alpha$  for  $\alpha \in \mathcal{I}_{n+1}$ . If  $i \leq \alpha$  then  $x_i \cdot Z_\alpha = Z_{i \cdot \alpha}$  by condition 2.



If  $i > \alpha$  then there exists  $\beta \in \mathcal{I}_n$  and  $j \in I$  with  $\alpha = j \cdot \beta$  such that  $i > j$  and  $j \leq \beta$ . If condition 3 was to hold for  $\varphi_{n+1}$  it follows that

$$x_i.Z_\alpha = x_i.x_j.Z_\beta = x_j.x_i.Z_\beta + [x_i, x_j].Z_\beta. \quad (6)$$

Because  $\beta \in \mathcal{I}_n$  the term  $[x_i, x_j].Z_\beta$  in the above sum is already defined. Because  $x_i.Z_\beta \equiv Z_{i.\beta} \pmod{V_{(n)}}$  there exists some  $Y \in V_{(n)}$  with  $x_i.Z_\beta = Z_{i.\beta} + Y$ . Let  $\gamma \in \mathcal{I}_{n+1}$  be defined by taking  $\beta$  and inserting  $i$  at the right position. Then  $Z_\gamma = Z_{i.\beta}$ . Because  $j < i$  and  $j \leq \beta$  it follows that  $j \leq \gamma$  and therefore

$$x_j.Z_{i.\beta} = x_j.Z_\gamma = Z_{i.\gamma} = Z_{i.(j.\beta)} = Z_{i.\alpha}$$

Because every summand in

$$\begin{aligned} x_i.Z_\alpha &= x_j.x_i.Z_\beta + [x_i, x_j].Z_\beta = x_j.(Z_{i.\beta} + Y) + [x_i, x_j].Z_\beta \\ &= Z_{i.\alpha} + x_j.Y + [x_i, x_j].Z_\beta \end{aligned} \quad (7)$$

is defined it follows that  $\varphi_{n+1}$  is uniquely defined.

Conditions 1 and 2 hold for  $\varphi_{n+1}$  by construction. Condition 4 holds for  $i \leq \alpha$  by condition 2 and for  $i > \alpha$  by (7) because  $x_j.Y \in V_{(n+1)}$  and  $[x_i, x_j].Z_\beta \in V_{(n+1)}$ .

It remains to check Condition 3 for  $\varphi_{n+1}$ , i.e. when  $i, j \in I$  and  $\alpha \in \mathcal{I}_n$ . For  $i = j$  this follows from the Lie bracket being alternating. Suppose that  $i \neq j$ . By the Lie bracket is antisymmetric it can be w.l.o.g. assumed that  $i < j$ .

If  $i \leq \alpha$  then  $x_j.x_i.Z_\alpha$  is defined above by using (6) (where  $\beta$  has to be replaced by  $\alpha$  and  $i$  and  $j$  have to be switched), hence condition 2 holds in this case by construction. Notice that if  $j \leq \alpha$  then also  $i \leq \alpha$  because  $i < j$ .

Hence the only case left is  $i \not\leq \alpha$ . By the above it then follows that also  $j \not\leq \alpha$ . As this cannot happen for  $n = 0$  it can be w.l.o.g. assumed that  $n \geq 1$ . Let  $k \in I$  and  $\beta \in \mathcal{I}_{n-1}$  with  $\alpha = k \cdot \beta$ . Because condition 3 holds for  $\varphi_n$  it follows that

$$\begin{aligned} x_i.x_j.Z_\alpha &= x_i.x_j.x_k.Z_\beta = x_i.(x_k.x_j.Z_\beta + [x_j, x_k].Z_\beta) \\ &= x_i.x_k.x_j.Z_\beta + x_i.[x_j, x_k].Z_\beta \end{aligned}$$

Because  $k < j$  and  $k \leq \beta$  it follows from the previous discussed cases that

$$x_i.x_k.(x_j.Z_\beta) = x_k.x_i.(x_j.Z_\beta) + [x_i, x_k].(x_j.Z_\beta).$$

Combining the above results in the equality

$$x_i.x_j.Z_\alpha = x_k.x_i.x_j.Z_\beta + [x_i, x_k].x_j.Z_\beta + x_i.[x_j, x_k].Z_\beta$$

By switching  $i$  and  $j$  in the above calculations it also follows that

$$x_j.x_i.Z_\alpha = x_k.x_j.x_i.Z_\beta + [x_j, x_k].x_i.Z_\beta + x_j.[x_i, x_k].Z_\beta$$

By using that condition 3 holds for  $\varphi_n$  it follows from these two equalities and the Jacobi identity that

$$\begin{aligned} x_i.x_j.Z_\alpha - x_j.x_i.Z_\alpha &= x_k.x_i.x_j.Z_\beta + [x_i, x_k].x_j.Z_\beta + x_i.[x_j, x_k].Z_\beta \\ &\quad - x_k.x_j.x_i.Z_\beta - [x_j, x_k].x_i.Z_\beta - x_j.[x_i, x_k].Z_\beta \\ &= x_k.[x_i, x_j].Z_\beta + [[x_i, x_k], x_j].Z_\beta + [x_i, [x_j, x_k]].Z_\beta \\ &= x_k.[x_i, x_j].Z_\beta - [x_k, [x_i, x_j]].Z_\beta = [x_i, x_j].x_k.Z_\beta = [x_i, x_j].Z_\alpha. \quad \square \end{aligned}$$

This finishes the proof.  $\square$

**Theorem 1.5.36** (PBW (abstract version)). *Let  $\mathfrak{g}$  be a Lie algebra over  $k$  and denote by  $\pi$  the canonical projection*

$$\pi: T(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n \quad \text{for all } x_1, \dots, x_n \in \mathfrak{g}.$$

*Then the two homomorphisms of graded  $k$ -algebras  $\text{gr}(\pi): T(\mathfrak{g}) \rightarrow \text{gr}(\mathcal{U}(\mathfrak{g}))$  and*

$$\pi': T(\mathfrak{g}) \rightarrow S(\mathfrak{g}), \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n \quad \text{for all } x_1, \dots, x_n \in \mathfrak{g}$$

*have the same kernel and thus induce an isomorphism of graded algebras*

$$\varphi: S(\mathfrak{g}) \rightarrow \text{gr}(\mathcal{U}(\mathfrak{g})), \quad x_1 \cdots x_n \mapsto [x_1 \cdots x_n] \quad \text{for all } x_1, \dots, x_n \in \mathfrak{g}. \quad (8)$$

**Remark 1.5.37.** Notice that the two multiplications in (8) live in different  $k$ -algebras.

**Proposition 1.5.38.** *The concrete version and the abstract versions of the PBW-theorem are equivalent.*

*Proof.* For  $x, y \in \mathfrak{g}$  it follows from the definition of  $\text{gr}(\pi)$  that

$$\text{gr}(\pi)(x \otimes y - y \otimes x) = [xy - yx] \in \text{gr}(\mathcal{U}(\mathfrak{g}))_2,$$

with representative  $xy - yx \in \mathcal{U}(\mathfrak{g})_{(2)}$ . By the definition of  $\mathcal{U}(\mathfrak{g})$  it follows that already  $xy - yx = [x, y] \in \mathcal{U}(\mathfrak{g})_{(1)}$ . Hence it follows for the residue class of  $xy - yx$  in  $\text{gr}(\mathcal{U}(\mathfrak{g}))_2 = \text{gr}(\mathcal{U}(\mathfrak{g}))_{(2)} / \text{gr}(\mathcal{U}(\mathfrak{g}))_{(1)}$  that  $[xy - yx] = [[x, y]] = [0] = 0$ . Hence  $\text{gr}(\pi)(x \otimes y - y \otimes x) = 0$ .

As the kernel of  $\pi'$  is generated by the element  $x \otimes y - y \otimes x$  with  $x, y \in \mathfrak{g}$  it follows that  $\pi'$  factorizes through a homomorphism of graded  $k$ -algebras  $\varphi$  as in Theorem 1.5.36.

(concrete  $\Rightarrow$  abstract) The algebra  $\mathcal{U}(\mathfrak{g})$  has a basis  $(x_\alpha \mid \alpha \in \mathcal{I})$ . It follows that  $\text{gr}(\mathcal{U}(\mathfrak{g}))_n$  has a basis given by the residue classes  $([x_\alpha] \mid \alpha \in \mathcal{I}_n)$ . The linear subspace  $S(\mathfrak{g})_n$  has a basis  $(x_1 \cdots x_n \mid (i_1, \dots, i_n) \in \mathcal{I}_n)$  which is mapped by  $\varphi_n$  to the above basis of  $\text{gr}(\mathcal{U}(\mathfrak{g}))_n$ . Hence  $\varphi_n$  is an isomorphism for every  $n \in \mathbb{N}$ , which is why  $\varphi$  is an isomorphism.

(abstract  $\Rightarrow$  concrete) As  $(x_\alpha \mid \alpha \in \mathcal{I})$  generates  $\mathcal{U}(\mathfrak{g})$  as a vector spaces by Lemma 1.5.35 it suffices to show that this collection is linearly independent. Suppose otherwise. Then there exists some minimal  $n \in \mathbb{N}$  such that  $(x_\alpha \mid m \leq n, \alpha \in \mathcal{I}_m)$  is linearly dependent. Hence there exists a non-trivial linear combination

$$0 = \sum_{m=0}^n \sum_{\alpha \in \mathcal{I}_m} \lambda_\alpha x_\alpha \quad \text{where } \lambda_\alpha = 0 \text{ for all but finitely many } \alpha \in \bigcup_{m=0}^n \mathcal{I}_m.$$

From this it follows that

$$0 = \sum_{m=0}^n \sum_{\alpha \in \mathcal{I}_m} \lambda_\alpha x_\alpha \equiv \sum_{\alpha \in \mathcal{I}_n} \lambda_\alpha x_\alpha \pmod{\mathcal{U}(\mathfrak{g})_{(n-1)}}$$

and hence that the equality  $\sum_{\alpha \in \mathcal{I}_n} \lambda_\alpha [x_\alpha] = 0$  holds in  $\text{gr}(\mathcal{U}(\mathfrak{g}))_n = \mathcal{U}(\mathfrak{g})_{(n)}/\mathcal{U}(\mathfrak{g})_{(n-1)}$ . By the minimality of  $n$  it follows that this is a non-trivial linear combination in  $\text{gr}(\mathcal{U}(\mathfrak{g}))_n$ , so  $([x_\alpha] \mid \alpha \in \mathcal{I}_n)$  in  $\text{gr}(\mathcal{U}(\mathfrak{g}))_n$  is linearly dependent.

By assumption  $\varphi$  is a homomorphism of graded  $k$ -algebras and therefore  $\varphi_n$  is an isomorphism of vector spaces. As the basis  $(x_{i_1} \cdots x_{i_n} \mid (i_1, \dots, i_n) \in \mathcal{I}_n)$  of  $S(\mathfrak{g})_n$  is mapped by  $\varphi_n$  bijectively to  $([x_\alpha] \mid \alpha \in \mathcal{I}_n)$  it follows that this is a basis of  $\text{gr}(\mathcal{U}(\mathfrak{g}))_n$ , contradicting the linear dependency.  $\square$

### 1.5.3. Casimir elements and operators

For this subsection we additionally assume that  $\mathfrak{g}$  is finite-dimensional. We also fix some bilinear form  $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  which is associative and non-degenerate.

**Definition 1.5.39.** Let  $\varphi_1: \mathfrak{g} \otimes \mathfrak{g}^* \rightarrow \text{End}_k(\mathfrak{g})$  and  $\varphi_2: \mathfrak{g} \rightarrow \mathfrak{g}^*$  be the isomorphisms of vector spaces defined by

$$\varphi_1(x \otimes \phi)(y) = \phi(y)x \quad \text{and} \quad \varphi_2(x) = \beta(x, \cdot) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } \phi \in \mathfrak{g}^*.$$

Then the image of 1 under the map

$$k \xrightarrow{\lambda \mapsto \lambda \text{id}_{\mathfrak{g}}} \text{End}_k(\mathfrak{g}) \xrightarrow{\varphi_1^{-1}} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\text{id}_{\mathfrak{g}} \otimes \varphi_2^{-1}} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{x \otimes y \mapsto xy} \mathcal{U}(\mathfrak{g}) \quad (9)$$

is called the *Casimir element of  $\beta$*  and denoted by  $C_\beta$ .

**Lemma 1.5.40.** *The Casimir element  $C_\beta$  is central in  $\mathcal{U}(\mathfrak{g})$ , i.e.*

$$xC_\beta = C_\beta x \quad \text{for every } x \in \mathcal{U}(\mathfrak{g}).$$

*Proof.* Let  $\varphi_1$  and  $\varphi_2$  as in Definition 1.5.39.

Because  $\mathcal{U}(\mathfrak{g})$  is generated by  $\mathfrak{g}$  as a  $k$ -algebra it suffices to show  $C_\beta$  commutes with every  $x \in \mathfrak{g}$ . Hence it is to show that

$$[x, C_\beta] = 0 \quad \text{for every } x \in \mathfrak{g},$$

where  $[\cdot, \cdot]$  denotes the Lie bracket in  $\mathcal{U}(\mathfrak{g})$ . To see this notice that in (9) every map is a homomorphism of representations of  $\mathfrak{g}$ , where  $\mathfrak{g}$  acts trivially on  $k$ , i.e.  $x \cdot \lambda = 0$  for every  $x \in \mathfrak{g}$  and  $\lambda \in k$ .

That the first map  $k \rightarrow \text{End}_k(\mathfrak{g})$  is a homomorphism of representations follows from the fact that  $\mathfrak{g}$  acts trivially on  $k$  and also trivially on the one-dimensional subspace  $k \text{id}_{\mathfrak{g}} \subseteq \text{End}_k(\mathfrak{g})$ .

That  $\varphi_1$  is an isomorphism of representations is known from Proposition 1.2.14.

That the third map  $\mathfrak{g} \otimes \mathfrak{g}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  is a homomorphism of representations follows from Proposition 1.2.14, because the identity  $\text{id}_{\mathfrak{g}}$  is a homomorphism of representations and the isomorphism  $\varphi_2$  is one by the associativity of  $\beta$ , as seen in Lemma 1.4.3.

That the fourth map  $\psi: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g}), x \otimes y \mapsto xy$  is a homomorphism of representations follows from direct calculation, because for all  $x, y, z \in \mathfrak{g}$

$$\begin{aligned} \psi(x \cdot (y \otimes z)) &= \psi((x \cdot y) \otimes z + y \otimes (x \cdot z)) = (x \cdot y)z + y(x \cdot z) \\ &= [x, y]z + y[x, z] = xyz - yxz + yxz - yzx = xyz - yzx \\ &= [x, yz] = x \cdot (yz) = x \cdot \psi(y \otimes z). \end{aligned}$$

Because every map in (9) is a homomorphism of representations it follows that their composition  $\phi: k \rightarrow \mathcal{U}(\mathfrak{g})$  is also a homomorphism of representations. Definition 1.5.39 is then equivalent to  $\phi(1) = C_\beta$ . Because  $\mathfrak{g}$  acts trivially on  $k$  and  $\phi$  is a homomorphism of representations it follows that  $\mathfrak{g}$  also acts trivially on the span of  $C_\beta$ . In particular

$$0 = x.C_\beta = [x, C_\beta] \quad \text{for every } x \in \mathfrak{g}. \quad \square$$

**Lemma 1.5.41** (Casimir in coordinates). *Let  $x_1, \dots, x_n$  be a basis of  $\mathfrak{g}$  and  $x^1, \dots, x^n$  the dual basis of  $\mathfrak{g}$  with respect to  $\beta$ , i.e.  $\beta(x_i, x^j) = \delta_{ij}$  for all  $i, j = 1, \dots, n$ . Then*

$$C_\beta = \sum_{i=1}^n x_i x^i.$$

*Proof.* Let  $\varphi_1$  and  $\varphi_2$  as in Definition 1.5.39. In (9) 1 is mapped to  $\text{id}_{\mathfrak{g}}$ , which is then mapped to  $\sum_{i=1}^n x_i \otimes x_i^*$ , where  $x_1^*, \dots, x_n^*$  denotes the dual basis of  $\mathfrak{g}^*$ . As  $\varphi_2(x^i) = x_i^*$  it follows that  $\sum_{i=1}^n x_i \otimes x_i^*$  is then mapped to  $\sum_{i=1}^n x_i \otimes x^i$ , which is then further mapped to the element  $\sum_{i=1}^n x_i x^i$  in  $\mathcal{U}(\mathfrak{g})$ .  $\square$

## 2. Semisimple Lie Algebras

### 2.1. Definition of semisimple Lie algebras

**Definition 2.1.1.** A Lie algebra  $\mathfrak{g}$  is called *semisimple* if it is the sum of finitely many simple ideals, i.e. if there exists ideals  $I_1, \dots, I_n \trianglelefteq \mathfrak{g}$  which are simple (as Lie algebras) such that  $\mathfrak{g} = I_1 \oplus \dots \oplus I_n$ .

**Theorem 2.1.2.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. The following are equivalent:

1.  $\mathfrak{g} \cong \mathfrak{g}_1 \times \dots \times \mathfrak{g}_r$  for some  $r \in \mathbb{N}$  and simple Lie algebras  $\mathfrak{g}_i, i = 1, \dots, r$ .
2.  $\mathfrak{g} = I_1 \oplus \dots \oplus I_s$  for some  $s \in \mathbb{N}$  and simple Ideals  $I_1, \dots, I_s \trianglelefteq \mathfrak{g}$ .
3. The Killing form  $\kappa$  of  $\mathfrak{g}$  is non-degenerate (which is equivalent to  $\text{rad } \kappa = 0$ ).
4.  $\mathfrak{g}$  has no nonzero solvable ideals (which is equivalent to  $\text{rad } \mathfrak{g} = 0$ ).
5.  $\mathfrak{g}$  has no nonzero abelian ideals.

*Proof.* (4  $\Rightarrow$  3) This directly follows from the fact that  $\text{rad } \kappa$  is a solvable ideal in  $\mathfrak{g}$ .

(3  $\Rightarrow$  2) The implication can be shown by induction over  $\dim \mathfrak{g}$ . If  $\dim \mathfrak{g} = 0$  then  $\mathfrak{g} = 0$  is the empty sum over zero simple ideals in  $\mathfrak{g}$ . Suppose that  $\dim \mathfrak{g} \geq 1$  and the implication holds for all smaller dimensions. If  $\mathfrak{g}$  is simple then there is nothing left to show. Otherwise  $\mathfrak{g}$  contains a non-trivial ideal  $I \trianglelefteq \mathfrak{g}$ , i.e.  $I \neq 0$  and  $I \neq \mathfrak{g}$ . Then

$$I^\perp := \{y \in \mathfrak{g} \mid \kappa(x, y) = 0 \text{ for every } x \in I\}$$

is an ideal in  $\mathfrak{g}$  and because  $\kappa$  is non-degenerate it follows that  $\dim \mathfrak{g} = \dim I + \dim I^\perp$ . Because  $I \cap I^\perp$  is an ideal in  $\mathfrak{g}$  it also follows that

$$\kappa_{I \cap I^\perp}(x, y) = \kappa(x, y) = 0 \quad \text{for all } x, y \in I \cap I^\perp.$$

By Cartan's criterion  $I \cap I^\perp$  is a solvable ideal in  $\mathfrak{g}$ , from which it follows from the previous implication (4  $\Rightarrow$  3) that  $I \cap I^\perp = 0$ . Therefore  $\mathfrak{g} = I \oplus I^\perp$ , where  $I$  and  $I^\perp$  are proper ideals in  $\mathfrak{g}$ . By Lemma 1.4.14 the Killing form  $\kappa$  is given by the sum of the Killing forms  $\kappa_{I_1}$  and  $\kappa_{I_2}$ . As  $\kappa$  is non-degenerate it follows that the same goes for  $\kappa_{I_1}$  and  $\kappa_{I_2}$ . Hence by induction hypothesis both  $I_1$  and  $I_2$  are the sum of simple ideals  $I_1 = J_1 \oplus \dots \oplus J_r$  and  $I_2 = K_1 \oplus \dots \oplus K_s$ . It follows that

$$\mathfrak{g} = I_1 \oplus I_2 = J_1 \oplus \dots \oplus J_r \oplus K_1 \oplus \dots \oplus K_s$$

is a decomposition into simple ideals.

(2  $\Rightarrow$  1) Follows from  $\mathfrak{g} = I_1 \oplus \cdots \oplus I_s \cong I_1 \times \cdots \times I_s$ .

(1  $\Rightarrow$  4) For each  $i = 1, \dots, r$  let  $\pi_i: \mathfrak{g} \rightarrow \mathfrak{g}_i$  be the canonical projection and let  $I \trianglelefteq \mathfrak{g}$  be a solvable ideal. Then for any  $i = 1, \dots, r$  the image  $\pi_i(I) \subseteq \mathfrak{g}_i$  is a solvable ideal. Because  $\mathfrak{g}_i$  is simple it follows that  $\pi_i(I) = 0$  for every  $i = 1, \dots, n$ . Hence  $I = 0$ .

(4  $\Rightarrow$  5) This directly follows from the fact that every abelian ideal is solvable.

(5  $\Rightarrow$  4) Suppose that there exists a nonzero solvable ideal  $I \trianglelefteq \mathfrak{g}$ . Then let  $i \geq 0$  such that  $I^{(i+1)} = 0$  but  $I^{(i)} \neq 0$ . Then  $I^{(i)}$  is a nonzero abelian ideal in  $\mathfrak{g}$ .  $\square$

**Corollary 2.1.3.** *Let  $\mathfrak{g}$  be a finite-dimensional, semisimple Lie algebra. Then the map*

$$\mathfrak{g} \rightarrow \mathfrak{g}^*, \quad x \mapsto \kappa(x, \cdot)$$

*is an isomorphism of representations of  $\mathfrak{g}$ .*

*Proof.* The statement follows from Corollary 1.4.4 because  $\kappa$  is non-degenerate.  $\square$

## 2.2. Theorem of Weyl

### 2.2.1. Casimir operators

# Appendices

# **A. Hopf algebras**

## **A.1. Coalgebras**



## B. Schur's Lemma

Unless otherwise noted  $k$  always is some arbitrary field. Whenever we talk about a ring (resp.  $k$ -algebra) we always mean an associative and unitary one, and homomorphisms of rings (resp.  $k$ -algebras) are required to respect the unit. We assume that the reader is familiar with the definition of a module over a ring notion of a submodules. By an (left)  $R$ -module  $M$  over a ring  $R$  we always mean an unital module, i.e.  $1 \cdot m = m$  for every  $m \in M$ .

### B.1. Classic version

**Definition B.1.1.** Let  $M$  be a module over a ring  $R$ . Then  $M$  is called *simple* or *irreducible* if  $M$  contains precisely two submodules. Equivalently  $M$  is nonzero and its only submodules are the *trivial* ones, namely  $0$  and  $M$  itself.

**Lemma B.1.2** (Schur). *Let  $R$  be a ring and  $M$  a simple module over  $R$ . Then any endomorphism of modules  $f: M \rightarrow M$  is either zero or an isomorphism. In particular  $\text{End}_R(M)$  is a skew field.*

*Proof.* As  $M$  is nonzero  $f$  cannot be zero and an isomorphism at the same time. If  $f \neq 0$  then  $\ker f$  is a proper submodule of  $M$  and  $\text{im } f$  is a nonzero submodule of  $M$ , so  $\ker f = 0$  and  $\text{im } f = M$  because  $M$  is simple.  $\square$

**Corollary B.1.3.** *Let  $M$  be an  $A$ -module over a  $k$ -algebra  $A$ . Then  $\text{End}_A(M)$  is a division algebra over  $k$ .*

**Lemma B.1.4.** *Let  $D$  be a division algebra over an algebraically closed field  $k$ . If  $x \in D$  is algebraic over  $k$  then already  $x \in k$ .*

*Proof.* Let  $P \in k[T]$  be nonzero with  $P(x) = 0$ . W.l.o.g.  $P$  can be assumed to be monic. Because  $k$  is algebraically closed there exist  $\alpha_1, \dots, \alpha_r \in k$  with  $P = \prod_{i=1}^r (x - \alpha_i)$ . Because  $0 = P(x) = c \prod_{i=1}^n (x - \alpha_i)$  and  $D$  is a skew field it follows that  $x = \alpha_i$  for some  $i$  and therefore  $x \in k$ .  $\square$

**Corollary B.1.5.** *Let  $k$  be an algebraically closed field and  $L$  a finite-dimensional division algebra over  $k$ . Then  $L = k$ .*

*Proof.* Let  $x \in L$ . Because  $L$  is finite-dimensional over  $k$  there exists some  $n \geq 1$  such that  $1, x, x^2, \dots, x^n$  are linearly dependent over  $k$ . Therefore there exist some  $a_0, a_1, \dots, a_n \in k$  such that  $a_0 + a_1x + \dots + a_nx^n = 0$  is a non-trivial linear combination. Then  $P = \sum_{i=0}^n a_iT^i \in k[T]$  is nonzero with  $P(x) = 0$ , so  $x$  is algebraic over  $k$ . From Lemma B.1.4 it follows that  $x \in k$ .  $\square$

**Corollary B.1.6** (Schur, classic Version). *Let  $k$  be an algebraically closed field and  $M$  a simple  $A$ -module for a  $k$ -algebra  $A$ . If  $M$  is finite-dimensional over  $k$  then  $\text{End}_A(M) = k$ , i.e. every module endomorphism of  $M$  is given by multiplication with a scalar.*

**Corollary B.1.7.** *Let  $\mathfrak{g}$  be a Lie algebra over an algebraically closed field  $k$  and  $V$  a irreducible and finite-dimensional representation of  $\mathfrak{g}$ . Then  $\text{End}_{\mathfrak{g}}(V) = k$ , i.e. every endomorphism of  $V$  as a representation of  $\mathfrak{g}$  is given by an multiplication with some scalar.*

*Proof.* Take  $V$  as a simple module over the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  and apply Corollary B.1.6.  $\square$

## B.2. Generalization by Dixmier

**Definition B.2.1.** Let  $V$  be a vector space over a field  $k$ . An endomorphism  $\varphi \in \text{End}_k(V)$  is called *algebraic* over  $k$  there exists some nonzero polynomial  $P \in k[T]$  mit  $P(\varphi) = 0$ .

**Lemma B.2.2.** *Let  $k$  be an algebraically closed field,  $V$  a vector space over  $k$  and  $D \subseteq \text{End}_k(V)$  a division algebra over  $k$ . If  $\varphi \in D$  is algebraic over  $k$  then  $\varphi = \alpha \text{id}_V$  for some  $\alpha \in k$ .*

*Proof.* This follows directly from Lemma B.1.4.  $\square$

**Corollary B.2.3.** *Let  $k$  be an algebraically closed field,  $A$  a  $k$ -algebra and  $M$  a simple  $A$ -module. If  $\varphi \in \text{End}_A(M)$  is algebraic then  $\varphi = \alpha \text{id}_M$  for some  $\alpha \in k$ .*

*Proof.* This follows directly from Lemma B.2.2 because  $\text{End}_A(M) \subseteq \text{End}_k(M)$  is a division algebra over  $k$  by Lemma B.1.2.  $\square$

The following Proposition traces back to [Dix63]. (At least this is what I found on the web — I could not find the original article, nor would I be able to read it (as it was apparently written in French)).

**Proposition B.2.4** (Dixmier). *Let  $M$  be a simple  $A$ -module for a  $k$ -algebra  $A$ , such that  $\dim_k M > \text{card } k$ . Then every  $\varphi \in \text{End}_A(M)$  is algebraic over  $k$ .*

*Proof.* Suppose that there exists some  $\varphi \in \text{End}_A(M)$  which is not algebraic over  $k$ . Then the kernel of the map

$$\iota: k[T] \rightarrow \text{End}_A(M), \quad P \mapsto P(\varphi)$$

is zero, hence  $\iota$  is an inclusion of  $k[T]$  into  $\text{End}_A(M)$ , which is a skew field by Lemma B.1.2. It follows That  $\iota$  can be extended to a well-defined inclusion

$$\theta: k(T) \rightarrow \text{End}_A(M), \quad \frac{P}{Q} \mapsto P(\varphi)Q(\varphi)^{-1}.$$

Hence  $M$  carries the structure of a  $k(T)$ -vector space with

$$\frac{P}{Q} \cdot m = P(\varphi)Q(\varphi)^{-1}(m) \quad \text{for every } \frac{P}{Q} \in k(T) \text{ and } m \in M.$$

As  $M$  is a nonzero  $k(T)$ -vector space it follows that  $\dim_k M \geq \dim_k k(T)$ . To see this notice that if  $L/k$  is any field extension and  $V$  a nonzero  $L$ -vector space then there exists an inclusion  $L \hookrightarrow V$  of  $L$ -vector spaces. This is then also an inclusion of  $k$ -vector spaces, which is why  $\dim_k V \geq \dim_k L$ . The statement follows with  $L = k(T)$  and  $V = M$ . (This is a straightforward generalization of the fact that every complex nonzero vector space is at least twodimensional as a real vector space.) Since  $(1/(T-a))_{a \in k}$  is a family of elements of  $k(T)$  which is linearly independent over  $k$  it also follows that  $\dim_k k(T) \geq \text{card } k$ .

Putting the above observations together it follows that

$$\dim_k M \geq \dim_k k(T) \geq \text{card } k,$$

contradicting the assumption that  $\text{card } k > \dim_k M$ . □

**Corollary B.2.5.** *Let  $k$  be an algebraically closed field,  $A$  a  $k$ -algebra and  $M$  a simple  $A$ -module. If  $\text{card } k > \dim_k M$  then  $\text{End}_A(M) = k$ .*

*Proof.* This is a combination of Corollary B.2.3 and Proposition B.2.4. □

**Corollary B.2.6.** *Let  $\mathfrak{g}$  be a Lie algebra over an algebraically closed field  $k$  and  $V$  an irreducible representation of  $\mathfrak{g}$  with  $\text{card } k > \dim_k V$ . Then  $\text{End}_{\mathfrak{g}}(V) = k$ .*

*Proof.* Take  $V$  as a simple module over the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  and apply Corollary B.2.5. □

**Example B.2.7.** Let  $\mathfrak{g}$  be complex Lie algebra and  $V$  an irreducible representation of  $\mathfrak{g}$  of countable dimension. Then  $\text{End}_{\mathfrak{g}}(V) = \mathbb{C}$ .

**Remark B.2.8.** The requirement that  $\text{card } k > \dim_k M$  in Corollary B.2.5 can not be dropped without adding some other restraints. To see this take  $k := \overline{\mathbb{Q}}$  as well as  $A = M = \overline{\mathbb{Q}}(T)$ . Then  $\dim_k M = \text{card } k$  and  $\text{End}_A(M) = \text{End}_{\overline{\mathbb{Q}}(T)}(\overline{\mathbb{Q}}(T)) = \overline{\mathbb{Q}}(T)$ .

## B.3. Generalization by Quillen

The following Proposition is due to [Qui69].

**Proposition B.3.1** (Quillen). *Let  $k$  be a field and  $A$  a filtered  $k$ -algebra, such that  $\text{gr } A$  is finitely generated and commutative as a  $k$ -algebra. If  $M$  is a simple  $A$ -module then every  $\varphi \in \text{End}_A(M)$  is algebraic over  $k$ .*

**Corollary B.3.2.** *Let  $\mathfrak{g}$  be finite-dimensional Lie algebra over an algebraically closed field  $k$  and  $V$  as irreducible representation of  $\mathfrak{g}$ . Then  $\text{End}_{\mathfrak{g}}(V) = k$ .*

*Proof.* Take  $V$  as a simple module over the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ . If  $x_1, \dots, x_n$  is a  $k$ -basis of  $\mathfrak{g}$  then by the abstract version of the PBW theorem

$$\mathrm{gr}\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g}) \cong k[x_1, \dots, x_n].$$

Applying Proposition B.3.1 to  $\mathcal{U}(\mathfrak{g})$  and  $V$  the statement follows from Corollary B.2.3.  $\square$

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